GROMOV–WITTEN THEORY OF PROJECTIVE BUNDLES

by

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ABSTRACT

In this dissertation, we will explore the Gromov–Witten theory of fibrations, especially with focus on projective bundles. It will be discussed from a few different perspectives and using different techniques. We hope our results might shed some light to the structure of the Gromov–Witten theory of fibrations, and hopefully offer some ideas for a few potential applications (e.g., the crepant transformation conjecture for ordinary flop and blow-up formulas).

For my parents.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

In this chapter, we would like to give a brief introduction and set up our notations.

1.1 Introduction

Gromov–Witten theory is a branch of enumerative geometry that studies a particular kind of deformation invariants (*Gromov–Witten invariants*) on smooth projective varieties or compact symplectic manifolds. It (virtually) counts the number of curves of a given degree that satisfy certain incidence conditions. Gromov–Witten theory was originated around the 1990s inspired by the development of string theory, and its rigorous foundations were later developed in both symplectic geometry and algebraic geometry. In this dissertation, we would like to study the Gromov–Witten theory of projective bundles and some related objects.

The appearance of projective bundles in Gromov–Witten theory is natural. It came up as key steps in the relative Gromov–Witten theory (e.g., [27]) and in flops and K-equivalence (crepant transformation) conjecture ([18–20]), among other things. Furthermore, in the study of the *functoriality* of the Gromov–Witten theory, the projective bundle also plays a role ([21, 22]). Namely, if one factors a projective morphism $f : X \to Y$ via embedding and smooth fibration

$$\begin{tikzcd} Y \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{P}_Y(E) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

(where E is a vector bundle over Y), one can study the general functoriality of Gromov–Witten theory by studying embedding (e.g., quantum Lefschetz hyperplane theorem) and projective bundles.

When the projective bundle splits (i.e., is the projectivization of a direct sum of line bundles), it naturally admits a nontrivial torus action on the total space that preserves fibers. This extra structure turns out to be effective in the study of its Gromov–Witten theory ([5, 9, 27]). This torus action fails to generalize if the projective bundle is non-split. The studies and applications of Gromov–Witten theory of non-split bundles exist, but they are relatively rare in literature. An example can be [18], where a degeneration argument is employed and has had some success in a limited form of *quantum splitting principle*. There are a few intriguing questions in the study of Gromov–Witten theory of non-split bundles which include the following.

Question 1. Let Y be a smooth variety and V be a vector bundle. Are the Gromov-Witten invariants of $\mathbb{P}(V)$ uniquely determined by that of the base Y and the total Chern class $c(V)$?

Although it appears to be basic, it has not been either proven or disproven in previous literature except simple cases (e.g., $Y = \mathbb{P}^1$).

The dissertation is organized as the following. In Chapter 2, we study the Gromov–Witten theory of split toric bundles (fibrations) which are generalizations of split projective bundles. The results in this chapter generalize the quantum Leray–Hirsch theorem in [20]. In Chapter 3, we study the Question 1 in the case when Y is a GKM manifold. Theorem 3.2.4 generalizes Brown's recursion technique in [5]. It also suggests a functoriality of equivariant Lagrangian cone in a wider range of objects. In Chapter 4, a different technique is applied and we successfully answer Question 1 in full generality. Its consequences include Theorem 4.5.1 which generalizes certain results in [16, 27].

1.2 Gromov–Witten invariants

In this section, we briefly review some basics and establish the Gromov–Witten theory (possibly twisted by a vector bundle) in both equivariant and nonequivariant contexts.

1.2.1 Moduli space of stable maps

Let C be a proper connected algebraic curve C of arithmetic genus g and X be a smooth projective variety. An *n-pointed genus-g prestable map* to X is a tuple $(C, x_1, \ldots, x_n, f)$ where C has at worst nodal singularities, $x_1, \ldots, x_n \in C^{sm}$ and f is a map from C to X. x_i is also called the *i-th marking*. A morphism between two prestable maps $(C, x_1, \ldots, x_n, f)$, $(C', x'_1, \ldots, f')$ is a map $\phi : C \to C'$ such that $\phi(x_i) = x'_i$ and $f = f' \circ \phi$. A prestable map is *stable* if its automorphism is finite. Concretely, let $(C, x_1, \ldots, x_n, f)$ be a prestable map and $\mathfrak{c} \subset C$ be an irreducible component of C. Let $\nu : \tilde{\mathfrak{c}} \to \mathfrak{c}$ be the normalization. Then C being stable is requiring that, for any irreducible component $\mathfrak{c}$,

$$\left|\{x \in \tilde{\mathfrak{c}} | \nu(x) \in C^{sing} \bigcup \{x_1, \ldots, x_n\}\}\right|$$

is larger than 2 if $g(\mathfrak{c}) = 0$, or larger than 0 if $g(\mathfrak{c}) = 1$. The curve class $f_*([C]) = \beta \in N_1(X)$ can be referred to as the *degree* of the stable map $(C, x_1, \ldots, x_n, f)$. The moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parametrizing n-pointed genus-g stable map to X with fixed degree β (can be precisely defined by representing a suitable moduli functor) exists as a projective Deligne-Mumford stack [3].

Let

$$\mathrm{ft}_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta)$$

be the map forgetting the last marked point. Under this map $\overline{\mathcal{M}}_{g,n+1}(X, \beta)$ can be viewed as the universal family over $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Furthermore, the evaluation map of the last marked point

$$\mathrm{ev}_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to X$$

serves as the universal stable map from the universal family over $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Let s_i be the section of ft_{n+1} given by sending a stable map to its i-th marking. The *i-th ψ-class* $\psi_i \in Pic(\overline{\mathcal{M}}_{g,n}(X, \beta))$ is defined as $s_i^* \omega_{\mathrm{ft}_{n+1}}$ where $\omega_{\mathrm{ft}_{n+1}}$ is the relative dualizing sheaf of ft_{n+1}. Roughly speaking, the fiber over $(C, x_1, \ldots, x_n, f)$ of the line bundle associated to ψ_i can be naturally identified as the cotangent space $T^*_{x_i} C$.

1.2.2 Perfect obstruction theory and virtual class

$\overline{\mathcal{M}}_{g,n}(X, \beta)$ is very hard to study in general because it does not have a lot of good properties (might be disconnected, non-reduced, have bad singularities, etc.). Instead, people constructed virtual (fundamental) classes on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ (e.g., [2, 25]) which enjoy better intersection theoretic properties. We would like to sketch the construction of perfect obstruction theory and the virtual class defined on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ according to [2] and [15, Appendix B].

There is a natural morphism

$$\tau : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$$

where $\mathfrak{M}_{g,n}$ is the Artin stack of n-pointed genus-g prestable curves. Let $L^\bullet_{\mathfrak{M}_{g,n}}$, $L^\bullet_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ be the cotangent complexes and $L^\bullet_\tau$ be the relative cotangent complex of τ. There is a distinguished triangle

$$\tau^* L^\bullet_{\mathfrak{M}_{g,n}} \to L^\bullet_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \to L^\bullet_\tau \to \tau^* L^\bullet_{\mathfrak{M}_{g,n}}[1].$$

$(R^\bullet(\mathrm{ft}_{n+1})_* \mathrm{ev}^*_{n+1} \Omega^\vee_X)^\vee$ has a natural morphism to $L^\bullet_\tau$ making it a relative perfect obstruction theory. Under the notation of [15, Appendix B], $(R^\bullet(\mathrm{ft}_{n+1})_* \mathrm{ev}^*_{n+1} \Omega^\vee_X)^\vee$ and $\tau^* L^\bullet_{\mathfrak{M}_{g,n}}$ are quasi-isomorphic to two-term complexes of vector bundles $B^\bullet = [B^{-1} \to B^0]$ and $A^\bullet = [A^0 \to A^1]$, respectively. There exists a complex $E^\bullet$ that completes the following diagram

$$\begin{tikzcd}
B^\bullet \arrow[r] \arrow[d] & A^\bullet[1] \arrow[r] \arrow[d] & E^\bullet[1] \arrow[r] \arrow[d] & B^\bullet[1] \arrow[d] \\
L^\bullet_\tau \arrow[r] & \tau^* L^\bullet_{\mathfrak{M}_{g,n}}[1] \arrow[r] & L^\bullet_{\overline{\mathcal{M}}_{g,n}(X,\beta)}[1] \arrow[r] & L^\bullet_\tau[1],
\end{tikzcd}$$

where the upper row is also a distinguished triangle. In order to see $E^\bullet$ is perfect, one needs to show that $\mathcal{H}^0(B^\bullet) \to \mathcal{H}^1(A^\bullet)$ is surjective. This amounts to showing that the map $R^1(\mathrm{ft}_{n+1})_* \left((\mathrm{ev}^*_{n+1} \Omega_X) \otimes \omega_{\mathrm{ft}_{n+1}} \right) \to R^1(\mathrm{ft}_{n+1})_* \left(\Omega_{\mathrm{ft}_{n+1}}(D) \otimes \omega_{\mathrm{ft}_{n+1}} \right)$ is surjective where D is the sum of divisors given by images of s_i. One only needs to worry about stable maps that have contracted components but the surjectivity can still be justified using the stability condition. We can see the morphism $E^\bullet \to L^\bullet_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ is a perfect obstruction theory. By suitably choosing $A^\bullet$ and $B^\bullet$, $E^\bullet$ can be made into a two-term complex of vector bundles.

Once a perfect obstruction theory is given, one can construct a virtual class following [2]. First of all, one has the fiber diagram

$$\begin{tikzcd}
C(E^\bullet) \arrow[r] \arrow[d] & (E^{-1})^\vee \arrow[d] \\
\mathfrak{C} \arrow[r] & {[(E^{-1})^\vee/(E^0)^\vee]},
\end{tikzcd}$$

where $\mathfrak{C}$ is the intrinsic normal cone of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and $E^\bullet = [E^{-1} \to E^0]$ a two-term complex of vector bundles resolving the perfect obstruction theory. In our case, the virtual class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} = 0^![C(E^\bullet)], \tag{1.1}$$

where 0 is the zero section of $(E^{-1})^\vee$ and $[C(E^\bullet)] \in A_*((E^{-1})^\vee)$.

Besides some good geometric properties (e.g., equi-dimensional, etc.), the merit of virtual classes also includes the virtual localization theorem [15]. It is an analog to Atiyah-Bott localization theorem [1], but it computes virtual fundamental classes instead of fundamental classes. Note that it holds even though $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is singular. The virtual localization formula will be recalled in section 1.3, and it will be used extensively in Chapter 3 and 4.

1.2.3 Nonequivariant Gromov-Witten theory

Let E be a vector bundle over X. Also let a $\mathbb{C}^*$ act on X trivially and on E by scaling (all irreducible sub-representations on a fiber are of character 1). Let λ be the corresponding equivariant parameter. There exists a two-term complex of vector bundles in $\overline{\mathcal{M}}_{g,n}(X,\beta)$

$$0 \to E^0_{g,n,\beta} \to E^1_{g,n,\beta} \to 0$$

such that the i-th cohomology is $R^i(\mathrm{ft}_{n+1})_* \mathrm{ev}^*_{n+1} E$ for $(i = 0, 1)$.

Let $\mathbb{C}^*$ act on $E^i_{g,n,\beta}$ by scaling as well. Write $E_{g,n,\beta}$ for the two-term complex $[E^0_{g,n,\beta} \to E^1_{g,n,\beta}]$ in $D^b(\overline{\mathcal{M}}_{g,n}(X,\beta))$. Define the equivariant Euler class

$$e_{\mathbb{C}^*}(E_{g,n,\beta}) = \frac{e_{\mathbb{C}^*}(E^0_{g,n,\beta})}{e_{\mathbb{C}^*}(E^1_{g,n,\beta})} \in H^*(\overline{\mathcal{M}}_{g,n}(X,\beta)) \otimes_{\mathbb{C}} \mathbb{C}[\lambda,\lambda^{-1}]. \tag{1.2}$$

As a result,

$$e_{\mathbb{C}^*}(E^1_{g,n,\beta}) = \lambda^{r_1} + c_1(E^1_{g,n,\beta})\lambda^{r_1-1} + \cdots + c_r(E^1_{g,n,\beta}) \in H^*(\overline{\mathcal{M}}_{g,n}(X,\beta)) \otimes_{\mathbb{C}} \mathbb{C}[\lambda], \tag{1.3}$$

where $r_1 = \mathrm{rank}(E^1_{g,n,\beta})$. Since elements in $H^*(\overline{\mathcal{M}}_{g,n}(X,\beta))$ are nilpotent, one easily sees that $e_{\mathbb{C}^*}(E_{g,n,\beta})$ is well-defined if we invert λ. Applying the same reason on $E^0_{g,n,\beta}$, we see $e_{\mathbb{C}^*}(E_{g,n,\beta})$ is invertible if λ can be inverted.

Define the *Gromov–Witten invariants twisted by E* to be

$$\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n \rangle^{X,tw,E}_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \frac{1}{e_{\mathbb{C}^*}(E_{g,n,\beta})} \cup \prod_{i=1}^n \psi_i^{k_i}\, \mathrm{ev}_i^*\,\alpha_i \in \mathbb{C}[\lambda,\lambda^{-1}]. \tag{1.4}$$

When $E = 0$ the zero sheaf, $E_{g,n,\beta} = 0$. We take $e_{\mathbb{C}^*}(E_{g,n,\beta}) = 1$ as a convention and call it *untwisted Gromov–Witten invariants* denoted by $\langle \cdots \rangle^X_{g,n,\beta}$.

Remark 1.2.1. In the more general framework [8], our twisted invariants are only the special case by choosing the multiplicative characteristic class to be $\frac{1}{e_{\mathbb{C}^*}(\cdot)}$. Throughout the dissertation, we use this characteristic class by default, and the use of a different one will be specially mentioned.

Twisted invariants already contain the information of untwisted invariants. To be more precise, we have the following lemma.

Lemma 1.2.2. *If the insertions are homogeneous and satisfy*

$$\sum_{i=1}^{n} k_i + \sum_{i=1}^{n} deg(\alpha_i) = dim([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}), \tag{1.5}$$

then

$$\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n \rangle_{g,n,\beta}^{X,tw,E} = \frac{\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n \rangle_{g,n,\beta}^{X}}{\lambda^r} \tag{1.6}$$

for some r*.*

Proof. One can expand to see that

$$\frac{1}{e_{\mathbb{C}^*}(E_{g,n,\beta})} = \frac{1}{\lambda^r}\left(1 + \cdots\right), \tag{1.7}$$

where $r = \operatorname{rank}(E^0_{g,n,\beta}) - \operatorname{rank}(E^1_{g,n,\beta})$. Each summand in $\cdots$ at the end involves cohomology classes of nonzero degrees. Since the insertion already agrees with the virtual dimension of the moduli space, only the leading term may produce a nonzero number. $\square$

1.2.4 Equivariant Gromov-Witten theory

Let $T := (\mathbb{C}^*)^m$ be an algebraic torus. When X admits a T action, we have parallel (equivariant) theories in cohomology, characteristic classes and Gromov–Witten theory. Equivariant theories under T action are studied for different purposes throughout the dissertation but all of which are based on the virtual localizaton technique summarized later in section 1.3.

Let E be a T-equivariant vector bundle on X. Recall we have $H_T^*(\text{point}) \cong \mathbb{C}[C(T)]$ where $C(T)$ is the group of characters of T. Once a basis is chosen, $\mathbb{C}[C(T)]$ can be written as a polynomial ring $\mathbb{C}[\lambda_1, \ldots, \lambda_m]$. The following notations are used throughout the chapter:

- $R_T = H_T^*(\{\text{point}\}) = \mathbb{C}[\lambda_1, \ldots, \lambda_m]$.

- S_T is the localization of R_T by the set of nonzero homogeneous elements.

Again, let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be the moduli of stable maps with curve class $\beta \in \mathrm{NE}(X)$. The T-action on X induces a T-action on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and we have an equivariant virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$. Again $E_{g,n,\beta}$ is isomorphic to a two-term complex of locally free sheaves

$$0 \to E^0_{g,n,\beta} \to E^1_{g,n,\beta} \to 0.$$

Suppose that the equivariant Euler class $e_T(E^0_{g,n,\beta})$ is invertible. Define

$$e_T(E_{g,n,\beta})^{-1} = \frac{e_T(E^1_{g,n,\beta})}{e_T(E^0_{g,n,\beta})}, \quad (\text{and } e_T(E_{g,n,\beta}) = \frac{e_T(E^0_{g,n,\beta})}{e_T(E^1_{g,n,\beta})} \text{ when } e_T(E^1_{g,n,\beta}) \text{ is invertible}).$$

Definition 1.2.3. Let ψ_i be the first chern class of the universal cotangent line bundle. Assume $e_T(E^0_{g,n,\beta})$ is invertible. Given $\alpha_1,\ldots,\alpha_n \in H^*_T(X)$, the *equivariant Gromov–Witten invariants twisted by E* are defined as

$$\langle \psi^{k_1}\alpha_1,\ldots,\psi^{k_n}\alpha_n\rangle^{X,eq,tw,E}_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \frac{1}{e_T(E_{g,n,\beta})} \prod_{i=1}^{n} \psi_i^{k_i}\,\mathrm{ev}_i^*\,\alpha_i. \tag{1.8}$$

Again the *untwisted equivariant Gromov–Witten invariants* are defined when $E = 0$, and denoted by $\langle \cdots \rangle^{X,eq}_{g,n,\beta}$.

Convention 1. *When the context is very clear, we might omit "eq" in the superscript to denote an equivariant Gromov–Witten invariant.*

Remark 1.2.4. $e_T(E_{g,n,\beta})$ is often not invertible. To remedy this situation, an auxiliary $\mathbb{C}^*$ action, scaling the fiber of E while acting trivially on X, is introduced. We have $R_{T\times\mathbb{C}^*} = R_T[x]$ where x is the equivariant parameter corresponding to the $\mathbb{C}^*$ action. Let $R_T[x,x^{-1}]\!]$ be the ring of Laurent series in x^{-1}. Let $(H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta))[x])_{\mathrm{loc}}$ be the localization of $H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta))[x]$ by inverting monic polynomials in x. It can be embedded in

$$H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta)) \otimes_{R_T} R_T[x,x^{-1}]\!]$$

by expanding the denominators. In this set-up,

$$e_{T\times\mathbb{C}^*}(E_{g,n,\beta})^{-1} = \frac{e_{T\times\mathbb{C}^*}(E^1_{g,n,\beta})}{e_{T\times\mathbb{C}^*}(E^0_{g,n,\beta})} \tag{1.9}$$

is defined in $(H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta))[x])_{\mathrm{loc}}$. The corresponding twisted invariants are defined when the Euler class and the insertions are embedded into $H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta)) \otimes_{R_T} R_T[x,x^{-1}]\!]$. The twisted invariants take values in $R_T[x,x^{-1}]\!]$.

$R_T[x,x^{-1}]\!]$ is sometimes too big to be useful, and we need to make some changes. To start with, $\dfrac{e_{T\times\mathbb{C}^*}(E^1_{g,n,\beta})}{e_{T\times\mathbb{C}^*}(E^0_{g,n,\beta})}$ is an element in $(H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta))[x])_{\text{loc}}$. In section 3.3, we embed $\dfrac{e_{T\times\mathbb{C}^*}(E^1_{g,n,\beta})}{e_{T\times\mathbb{C}^*}(E^0_{g,n,\beta})}$ into $(H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta))[x])_{\text{loc}}\otimes_{R_T} S_T$, and take $x=0$ limit in $H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta))\otimes_{R_T} S_T$. The existence of the limit means that there are elements $\mathfrak{e}_0,\mathfrak{e}_1\in H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta))\otimes_{R_T} S_T$ with $\mathfrak{e}_0$ *invertible* such that

$$\mathfrak{e}_1 e_{T\times\mathbb{C}^*}(E^0_{g,n,\beta})-\mathfrak{e}_0 e_{T\times\mathbb{C}^*}(E^1_{g,n,\beta})\in xH^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta))[x]\otimes_{R_T} S_T.$$

The twisted invariants are defined by replacing $e_T(E_{g,n,\beta})^{-1}$ in Definition 1.2.3 with $\dfrac{\mathfrak{e}_1}{\mathfrak{e}_0}$. They take values in S_T.

For any ring R, $R[\![Q]\!]$ shall always be understood as the ring of Novikov variables, i.e., the ring of power series of $Q^\beta,\beta\in\mathrm{NE}(X)$ with coefficients in R. (See, e.g., [24, Chapter 1].) The notations $R[\![z,z^{-1}]$ and $R[z,z^{-1}]\!]$ in this chapter stand for the *ring* of formal Laurent series, with infinite powers in z and z^{-1}, respectively (and finite in the other).

1.2.5 Lagrangian cone formulation

Let us recall Givental's Lagrangian cone formalism and fix the notations. The readers are referred to [14] for details of Givental's construction and various properties of the Lagrangian cones.

Let $\mathcal{H}=H^*_T(X)\otimes_{R_T} S_T[z,z^{-1}]\!][\![Q]\!]$. There is a bilinear pairing $(,)_E$ induced by the Poincaré pairing

$$(\alpha,\beta)_E=\int_{[X]} e_T(E)^{-1}\alpha\cup\beta \tag{1.10}$$

on $H^*_T(X)$. Let $\mathcal{H}_+=H^*_T(X)\otimes_{R_T} S_T[z][\![Q]\!],\quad \mathcal{H}_-=z^{-1}H^*_T(X)\otimes_{R_T} S_T[\![z^{-1}]\!][\![Q]\!]$ $\mathcal{H}$ admits a *polarization* $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ and is indeed naturally identified as the cotangent bundle $T^*\mathcal{H}_+$. The canonical symplectic structure of $\mathcal{H}\cong T^*\mathcal{H}_+$ is the $S_T[\![Q]\!]$-bilinear form

$$\Omega(f(z),g(z))=\mathrm{Res}_{z=0}(f(-z),g(z))_E dz \tag{1.11}$$

for any $f,g\in\mathcal{H}$.

Givental's (twisted) Lagrangian cone is defined as the section of $T^*\mathcal{H}_+$ by the differential of genus-0 Gromov–Witten descendant potential, defined in a formal neighborhood at $-1z\in\mathcal{H}_+$.

More explicitly, let $\{\phi_i\}$ be a basis of R_T-module in $H^*_T(X)$ that induces a $\mathbb{C}$-basis in $H^*_T(X)/H^*_T(\{\text{point}\})$. Assume $\phi_0 = 1$ and let ϕ^i be the dual basis in $H^*_T(X) \otimes_{R_T} S_T$ with respect to $(,)_E$. We parametrize cohomology classes by taking $t_k = \sum_i t^i_k \phi_i$ with t^i_k formal variables. Let $t(z) = t_0 + t_1 z + \cdots + t_k z^k + \cdots$ be the formal parameters in $\mathcal{H}_+$, and $(\mathcal{H}, -1z)$ be the formal neighborhood at $-1z \in \mathcal{H}$. A general (formal) family of points $F \in (\mathcal{H}, -1z)$ in Givental's (twisted) Lagrangian cone $\mathcal{L}_{X,E} \subset (\mathcal{H}, -1z)$ is of the form

$$\begin{aligned} F(-z,t) &= -1z + t(z) + \sum_{\beta \in \mathrm{NE}(X)_{\mathbb{Z}}} \sum_{n=0}^{\infty} \sum_{j=1}^{N} \frac{Q^\beta}{n!} \langle \frac{\phi_j}{-z-\psi}, t(\psi), \ldots, t(\psi) \rangle^{X,E}_{0,n,\beta} \phi^j \\ &= -1z + t(z) + \sum_{\beta \in \mathrm{NE}(X)_{\mathbb{Z}}} \sum_{n=0}^{\infty} \frac{Q^\beta}{n!} (\mathrm{ev}_1)_* \left[\frac{e_T^{-1}(E_{0,n+1,d})}{-z-\psi_1} \prod_{i=2}^{n+1} \mathrm{ev}_i^* t(\psi_i) \right]. \end{aligned} \tag{1.12}$$

The definition of Lagrangian cone depends on the polarization. Different polarizations will be employed as necessary. See Convention 4.

Remark 1.2.5. Note that the various functions only exist in various suitably completed spaces of $\mathcal{H}$, rather than $\mathcal{H}$ itself. The explanation of "formal neighborhood" and a more rigorous definition of the theory can be made using formal schemes as in Appendix B of [7] where it is painstakingly spelled out (and in different guises in [24]).

1.3 Virtual localization

This section summarizes the localization technique under an arbitrary T-action. Let X be a smooth projective variety admitting an action by a torus $T = (\mathbb{C}^*)^m$. It induces an action of T on $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Let $\overline{\mathcal{M}}_\alpha$ be the connected components of the fixed loci in $\overline{\mathcal{M}}_{g,n}(X,\beta)^T$ labeled by α with the inclusion $i_\alpha : \overline{\mathcal{M}}_\alpha \to \overline{\mathcal{M}}_{g,n}(X,\beta)$. The virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ can be written as [15]

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} = \sum_\alpha (i_\alpha)_* \frac{[\overline{\mathcal{M}}_\alpha]^{vir}}{e_T(N^{vir}_\alpha)}, \tag{1.13}$$

where $[\overline{\mathcal{M}}_\alpha]^{vir}$ is constructed from the fixed part of the restriction of the perfect obstruction theory of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, and the virtual normal bundle N^{vir}_α is the moving part of the two-term complex in the perfect obstruction theory of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Note that in our situation, the indices α are the decorated graphs defined in the Definition 1.3.14.

1.3.1 General facts on torus action

In this subsection and the next, a few general facts about torus actions are stated, many without proofs. These facts are straightforward consequences of results in [4] and [28]. However, they are not expressed there in exactly the forms needed for our purpose and we have thus elected to collect them here.

Without further specification, by "invariant", we always mean T-invariant and an *irreducible invariant curve* on X is always assumed to be a proper reduced irreducible one-dimensional T-invariant subscheme of X.

$X^T \subset X$ denotes the fixed subscheme in X under the torus action by T. Let $Z_1, \ldots, Z_l$ be connected components of X^T. Let $N_j := N_{Z_j/X}$ be the normal bundle of Z_j in X and $\iota_j : Z_j \to X$ be the inclusion.

Recall that

$$C(T) = Hom_{\mathbb{C}^*}(T, \mathbb{C}^*) \tag{1.14}$$

is the group of characters of T. We have the decomposition of the normal bundle N_j on Z_j into eigensheaves

$$N_j = \bigoplus_{\chi \in C(T)} N_{j,\chi}. \tag{1.15}$$

Let $\rho : \mathbb{C}^* \to T$ be a 1-parameter subgroup.

Definition 1.3.1. Define (ρ, χ) to be the weight of the composition $\mathbb{C}^* \xrightarrow{\rho} T \xrightarrow{\chi} \mathbb{C}^*$.

Since N_j is of finite rank, for a generic 1-parameter subgroup ρ, we have $(\rho, \chi) = 0$ if and only if either $\chi = 0$ or $N_{j,\chi} = 0$.

Proposition 1.3.2. *There is a unique T-invariant reduced and irreducible closed subscheme Z_j^ρ such that*

1. $Z_j \subset Z_j^\rho$,
2. Z_j^ρ *is regular along* Z_j,
3. $N_{Z_j/Z_j^\rho} \cong \bigoplus_{(\rho,\chi)>0} N_{j,\chi}$.

This can be seen by first restricting the T-action to the 1-parameter subgroup determined by ρ. One can apply [4, Theorem 4.1] and take $Z_j^\rho := (X_j)^+$ (note that $(X_j)^+$ is under the notation of [4] when their G is our T and their $(X^G)_j$ is our Z_j). Notice that a point $x \in Z_j^\rho$ is characterized by $\lim_{\lambda\to 0} \rho(\lambda) \cdot x \in Z_j$. Since the group T is commutative, Z_j^ρ is also T-invariant.

Furthermore, by [4, Theorem 2.5], we are able to find a neighborhood U of Z_j in Z_j^ρ that is locally isomorphic to $(Z_j \cap U) \times V$ with V a $\mathbb{C}^*$ representation. We first of all use this decomposition to describe irreducible invariant curves in X. An irreducible invariant curve on X must be either contained in a Z_j, or as the closure of a one-dimensional orbit. Any one-dimensional orbit $\mathfrak{o}$ is isomorphic to $\mathbb{C}^*$ as it is a quotient of a torus. Suppose C intersects Z_j at p. If we fix the isomorphism $\mathfrak{o} \cong \mathbb{C}^*$ in a way that the limit toward 0 is p, then $T \to \mathfrak{o} \cong \mathbb{C}^*$ determines a character χ_p. We call χ_p the *(integral) character of an irreducible invariant curve near* p in this case. If the irreducible invariant curve is in X^T, we set $\chi_p = 0$.

Now given a ρ as in Proposition 1.3.2 and an irreducible invariant curve C with $p \in C \cap X^T$.

Proposition 1.3.3. *If $(\rho, \chi_p) > 0$, C is contained in Z_j^ρ.*

Proof. X can be covered by T-invariant affine open sets by [30]. Let U be such an open neighborhood of p and let $I_C \subset \mathbb{C}[U]$ be the ideal of $C \cap U \subset U$. Since U is T-invariant, $\mathbb{C}[U]$ is graded by the characters lattice $C(T)$, and I_C is a homogeneous ideal. Because $C \setminus \{p\}$ is isomorphic to $\mathbb{C}^*$ on which T acts by χ_p, $\mathbb{C}[U]/I_C \subset \mathbb{C}[C \setminus \{p\}] \cong \mathbb{C}[t, t^{-1}]$ where t is a homogeneous element graded by χ_p. Because the limit $t \to 0$ exists (which is p), $\mathbb{C}[U]/I_C$ has a nonempty $a\chi_p$-graded piece only if $a > 0$. In other words, the image of $\mathbb{C}[U]/I_C \subset \mathbb{C}[t, t^{-1}]$ in fact lies in $\mathbb{C}[t]$.

By the construction in [4], Z_j^ρ is cut out by homogeneous elements $u \in \mathbb{C}[U]$ graded by χ' such that $(\rho, \chi') < 0$. Now the image of u in $\mathbb{C}[U]/I_C$ must be zero, as $(\rho, \chi_p) > 0$ and the fact that $\mathbb{C}[U]/I_C$ has nonempty $a\chi_p$-graded piece only if $a > 0$. Therefore, any such t lies in I_C, i.e., C is a closed subscheme of Z_j^ρ. □

Corollary 1.3.4. *Let C be the closure of a one-dimensional orbit $\mathfrak{o} \cong \mathbb{C}^*$. The limits of $0, \infty$ land in different connected components Z_j, $Z_{j'}$ of X. Furthermore, in a neighborhood of $p \in Z_j \cap C$, the irreducible invariant curve C can be parameterized by t as $(c_1 t^{a_1}, \ldots, c_n t^{a_n})$.*

Proof. The first part is obvious. There is an open neighborhood U of p, such that $Z_j^\rho \bigcap U$ is some open set times a $\mathbb{C}^*$ representation V. Since C is invariant, $C \bigcap U \subset \{p\} \times V$. As in the previous proof, embed $\mathbb{C}[V]/I_C \hookrightarrow k[t]$. Let $x_1, \ldots, x_n$ be linear functions in $\mathbb{C}[V]$ that are homogeneous with respect to $\mathbb{C}^*$ action. The image of x_i in $\mathbb{C}[t]$ must be homogeneous. Thus, it is $c_i t^{a_i}$ for some $c_i \in \mathbb{C}, a_i \in \mathbb{Z}_{>0}$. □

In particular, we have

Corollary 1.3.5. *An irreducible invariant curve is homeomorphic to its normalization.*

1.3.2 T-invariant stable maps

We now proceed to study genus zero invariant stable maps. The contents of this subsection follow from results in [28] (and [6]).

Define $C(T)_{\mathbb{Q}} := C(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $f : (C, x_1, \ldots, x_n) \to X$ be an *invariant* stable map and C_0 an irreducible component of C. Let $p \in C_0$ be a point such that $f(p) \in X^T$.

Definition 1.3.6. Let $\chi_{f(p)}$ be the integral character of the irreducible invariant curve $f(C_0)$ at $f(p)$ and k be the degree of $f|_{C_0} : C_0 \to f(C_0)$. Define the *fractional character of* C_0 *at* p to be $\chi_p = \chi_{f(p)}/k \in C(T)_{\mathbb{Q}}$. In particular, when $f(C_0) \subset X^T$, χ is 0.

Remark 1.3.7. We would like to emphasize again that there is a big difference of definitions between the *integral character* at a fixed point on a one-dimensional orbit closure, and the *fractional character* at a point on the domain curve of an invariant stable map.

Proposition 1.3.8 ([28]). *The fractional character χ_p defined above is deformation invariant for* T-invariant *stable maps. More explicitly, let $f : (\mathcal{C}, x_0, \ldots, x_n) \to X$ be a family of invariant stable maps over a connected base S and $x_i : S \to \mathcal{C}$ the sections to the corresponding markings. Let $(C_p)_0$ be the unique irreducible component of the fiber C_p containing $x_0(p)$. Then the fractional character of $(C_p)_0$ at $x_0(p)$ does not depend on the choice of $p \in S$.*

Remark 1.3.9. In this proposition, stability is crucial. One can construct a counterexample for semistable curves by making a family of curves specialized into a curve with x_0 lying on a contracted component.

Proposition 1.3.10 ([28]). *Let $f : (C, x_1, \ldots, x_n) \to X$ be an invariant stable map. Let C_1, C_2*

be two irreducible components of C *such that* $p \in C_1 \cap C_2$ *is a node. The fractional characters of* C_1, C_2 *at* p *are denoted as* χ_1, χ_2 *,respectively. Suppose* C_1, C_2 *are not contracted under* f*, then the node* p *can be smoothened in a family of invariant stable maps only if* $\chi_1 + \chi_2 = 0$*.*

The condition $\chi_1 + \chi_2 = 0$ is consistent with [28, Definition 2.7(3)] and is a necessary condition for smoothability within a component of the fixed substack $\overline{\mathcal{M}}_{g,n}(X, \beta)^T$. In view of this, we make the following definition.

Definition 1.3.11. Notations are the same as the above proposition. x is said to be a node satisfying condition $(\star)$ if

$$(\star) \qquad \chi_1 \neq 0 \text{ and } \chi_1 + \chi_2 = 0.$$

We would like to introduce an important type of invariant stable maps whose moduli spaces are main building blocks in the components of $\overline{\mathcal{M}}_{g,n}(X, \beta)^T$, cf. [28, § 3] and [6, § 7.4].

Definition 1.3.12. Given two fixed loci $Z_j, Z_{j'}$, by an *unbroken map of a nonzero fractional character* $\chi \in C(T)_{\mathbb{Q}}$ *between* $Z_j, Z_{j'}$, we mean an invariant stable map with 2-markings $f : (C, x_+, x_-) \to X$ such that

1. C is a chain of $\mathbb{P}^1$, with irreducible components $C_1, \ldots, C_l$ such that $C_i \cap C_{i+1} \neq \emptyset$;
2. $x_+ \in C_1$, $x_- \in C_l$. Also $f(x_+) \in Z_j$, and $f(x_-) \in Z_{j'}$;
3. the fractional character of C at x_+ is χ, and all nodes in C satisfy $(\star)$.

Later, moduli spaces of unbroken maps are used to describe the fixed substack of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ under the torus action. We list all related definitions and notations below for later references.

Notation 1.

- Given Z_j, $Z_{j'}$ and a curve class $d \in \mathrm{NE}(X)$, $\overline{\mathcal{M}}_{j,j',\beta,\chi}$ denotes the closed substack of $\overline{\mathcal{M}}_{0,2}(X, \beta)$ parametrizing unbroken maps of fractional character χ between Z_j and $Z_{j'}$ whose curve class is d.

- Denote $\pi_{j,j',\beta,\chi} : \mathcal{C}_{j,j',\beta,\chi} \to \overline{\mathcal{M}}_{j,j',\beta,\chi}$ the universal family and $f_{j,j',\beta,\chi} : \mathcal{C}_{j,j',\beta,\chi} \to X$ the corresponding universal stable map.

- Let $[\overline{\mathcal{M}}_{j,j',\beta,\chi}]^{vir}$ be the virtual class induced by the restriction of the natural perfect obstruction theory from $\overline{\mathcal{M}}_{0,2}(X,\beta)$.

- There are two sections $s^{+}_{j,j',\beta,\chi}, s^{-}_{j,j',\beta,\chi} : \overline{\mathcal{M}}_{j,j',\beta,\chi} \to \mathcal{C}_{j,j',\beta,\chi}$ mapping an unbroken map $f : (C, x_+, x_-) \to X$ between $Z_j, Z_{j'}$ to the markings x_+ and x_-, respectively.

- $\psi_+ = (s^{+}_{j,j',\beta,\chi})^* \omega_{\pi_{j,j',\beta,\chi}}$, $\psi_- = (s^{-}_{j,j',\beta,\chi})^* \omega_{\pi_{j,j',\beta,\chi}}$, where $\omega_{\pi_{j,j',\beta,\chi}}$ is the relative dualizing sheaf.

- $\mathrm{ev}_+ = f_{j,j',\beta,\chi} \circ s^{+}_{j,j',\beta,\chi} : \overline{\mathcal{M}}_{j,j',\beta,\chi} \to Z_j$ and $\mathrm{ev}_- = f_{j,j',\beta,\chi} \circ s^{-}_{j,j',\beta,\chi}$.

- Let $X_{j,j',\beta,\chi} \subset X$ be the image of $f_{j,j',\beta,\chi}(\mathcal{C}_{j,j',\beta,\chi})$.

Example 1.3.13. Let $T = \mathbb{C}^*$ act on $\mathbb{P}^2$ by sending $[x_0; x_1; x_2]$ to $[\lambda^{-2}x_0; \lambda^{-1}x_1; x_2]$. By a slight abuse of notation, also let λ be the identity character of $\mathbb{C}^*$. Let $Z_0 = [1;0;0], Z_1 = [0;1;0], Z_2 = [0;0;1]$. The moduli of unbroken maps between Z_0 and Z_2 with character λ form a one-dimensional family that sweeps through $\mathbb{P}^2$. A generic member of such unbroken maps have irreducible domain curve, but a special one ($[a;b;0], a, b \in \mathbb{C}$) breaks in Z_1 with condition $(\star)$ satisfied.

Let l be the class of a line in $\mathbb{P}^2$. In this example, $\overline{\mathcal{M}}_{0,1,l,\lambda} = \{\mathrm{pt}\}, \overline{\mathcal{M}}_{0,2,l,\lambda} = \mathbb{P}(1,2)$. For $\overline{\mathcal{M}}_{0,2,l,\lambda/k}$, they parametrize degree k covers of unbroken maps between Z_0, Z_2. Although the coarse moduli spaces are the same, we have $\overline{\mathcal{M}}_{0,2,l,\lambda/k} = \mathbb{P}(k,2k)$.

1.3.3 Decorated graphs and moduli of invariant stable maps

We associate graphs to combinatorial types of fixed loci $\overline{\mathcal{M}}_{g,n}(X,\beta)^T \subset \overline{\mathcal{M}}_{g,n}(X,\beta)$, following the notations in [26, Definition 52].

Definition 1.3.14. A decorated graph $\vec{\Gamma} = (\Gamma, \vec{p}, \vec{\beta}, \vec{s}, \vec{\chi})$ for a genus-0, n-pointed, degree $\beta \in \mathrm{NE}(X)$ invariant stable map consists of the following data.

- Γ a finite connected graph, $V(\Gamma)$ the set of vertices and $E(\Gamma)$ the set of edges;

- $F(\Gamma) = \{(e,v) \in E(\Gamma) \times V(\Gamma) \mid v \text{ incident to } e\}$ the set of flags;

- the label map $\vec{p} : V(\Gamma) \to \{1, \ldots, l\}$;
- the degree map $\vec{\beta} : E(\Gamma) \cup V(\Gamma) \to \mathrm{NE}(X) \cup \bigcup_j \mathrm{NE}(Z_j)$, where $\vec{\beta}$ sends a vertex v to $\mathrm{NE}(Z_{\vec{p}(v)})$ and an edge e to $\mathrm{NE}(X)$;
- the genus map $\vec{g} : V(\Gamma) \to \mathbb{Z}_{\geq 0}$.
- the marking map $\vec{s} : \{1, 2, \ldots, n\} \to V(\Gamma)$ for $n > 0$;
- the fractional character map $\vec{\chi} : F(\Gamma) \to C(T)_{\mathbb{Q}}$.

They are required to satisfy the following conditions:

- $V(\Gamma), E(\Gamma), F(\Gamma)$ determine a connected graph.
- $\sum\limits_{e \in E(\Gamma)} \vec{\beta}(e) + \sum\limits_{v \in V(\Gamma)} \vec{\beta}(v) = \beta$.
- For any vertex v of valence 2, the two edges e_1, e_2 incident to v satisfy $\vec{\chi}((e_1, v)) + \vec{\chi}((e_2, v)) \neq 0$.

We associate a decorated graph $\vec{\Gamma}$ to an invariant stable map f in the following way.

Vertices:

- The connected components in $f^{-1}(X^T)$ are either curves or points. Assign a vertex v to a connected component $\mathfrak{c}_v$ in $f^{-1}(X^T)$ that is either a sub-curve of C, a smooth point in C, or a node in C that does *not* satisfy condition ($\star$).
- Define $\vec{p}(v) = j$ where $1 \leq j \leq l$ is the label such that $f(\mathfrak{c}_v) \in Z_j$.
- Define $\vec{\beta}(v) = [\mathfrak{c}_v] \in \mathrm{NE}(Z_j)$ the numerical class $f_*([\mathfrak{c}_v])$ (notice if $\mathfrak{c}_v$ is a point, $[\mathfrak{c}_v] = 0$).
- Define $\vec{g}(v)$ to be the genus of $\mathfrak{c}_v$.
- For $i = 1, \ldots, n$, define $\vec{s}(i) = v$ if $x_i \in \mathfrak{c}_v$.

Edges:

- Assign each component of $C \setminus \bigcup\limits_{v \in V(\Gamma)} \mathfrak{c}_v$ an edge e. Let $\mathfrak{c}_e$ be the closure of the corresponding component.

- We write $\vec{\beta}(e) = f_*[\mathfrak{c}_e] \in \mathrm{NE}(X)$.

Flags:

- $F(\Gamma) = \{(e,v) \in E(\Gamma) \times V(\Gamma) \mid \mathfrak{c}_e \cap \mathfrak{c}_v \neq \emptyset\}$
- Given a flag (e,v), $\mathfrak{c}_e \cap \mathfrak{c}_v$ must consist of a single point x. Let the fractional character of $\mathfrak{c}_e$ at x be χ. Define $\vec{\chi}((e,v)) = \chi$.

Remark 1.3.15. In the case of toric variety, for any invariant stable map, none of the nodes in $f^{-1}(X^T)$ can satisfy condition ($\star$). Thus, we reduce back to the traditional definition of decorated graphs in, for example, [26].

Convention 2. *To further shorten the expressions, we adopt the following convention when the decorated graph $\vec{\Gamma}$ is given in the context.*

- $p_v = \vec{p}(v)$,
- $s_i = \vec{s}(i)$,
- $\beta_v = \vec{\beta}(v)$, $\beta_\Gamma = \sum_{e \in E(\Gamma)} \vec{\beta}(e) + \sum_{v \in V(\Gamma)} \vec{\beta}(v)$,
- $\chi_{e,v} = \vec{\chi}((e,v))$.
- $g_v = \sum_{v \in V(\Gamma)} \vec{g}(v) + h^1(\Gamma)$ *where* $h^1(\Gamma)$ *is the dimension of the first singular cohomology of* Γ *as a graph.*

Since edge curves with nontrivial T-action can only degenerate to create nodes satisfying condition ($\star$), the associated graph is invariant under deformations.

Definition 1.3.16. Given a vertex $v \in V(\Gamma)$, define $E_v = \{e \in E(\Gamma) | (e,v) \in F(\Gamma)\}$ to be the set of edges containing v. Also define $S_v = \vec{s}^{-1}(v) \subset \{1, \ldots, n\}$. Define $\mathrm{val}(v) = |E_v|$ to be the valence of v, and $n_v = |S_v|$ to be the number of marked points on v.

An edge e must connect two vertices. We call the two vertices $v_1(e), v_2(e)$ whose order is arbitrary. Given a vertex v, we write $\overline{\mathcal{M}}_{0,E_v \cup S_v}(Z_j, d)$ instead of $\overline{\mathcal{M}}_{0,\mathrm{val}(v)+n_v}(Z_j, d)$ if we want to index the marked points by edges in E_v and markings in S_v. Under such notations, the corresponding evaluation map at marked point labeled by $e \in E_v$ is denoted by ev_e.

Given a decorated graph $\vec{\Gamma}$, one can construct the corresponding fixed component in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ out of fiber products of $\overline{\mathcal{M}}_{j,j',\beta,\chi}$ and $\overline{\mathcal{M}}_{0,n_i}(Z_j,d)$. Let $\vec{\Gamma}$ be a decorated graph. Define

$$\overline{\mathcal{M}}_{\vec{\Gamma}} = \left[\prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g_v,E_v\cup S_v}(Z_{p_v},d_v)\right] \times' \left[\prod_{e\in E(\Gamma)} \overline{\mathcal{M}}_{p_{v_1(e)},p_{v_2(e)},d_e,\chi_{e,v_1(e)}}\right], \tag{1.16}$$

where we make the convention that $\overline{\mathcal{M}}_{g,n}(Z_j,0) := Z_j$ for $n = 1,2$. Here the $\times'$ is the fiber product defined as follows. For any flag (e,v), let v' be the other vertex on e different from v. The $\times'$ identifies the evaluation map ev_e in $\overline{\mathcal{M}}_{g_v,E_v\cup S_v}(Z_{p_v},d_v)$ with ev_+ in $\overline{\mathcal{M}}_{p_v,p_{v'},d_e,\chi_{e,v_1(e)}}$.

Let $\mathrm{Aut}(\Gamma)$ be the automorphism of graphs. Note that automorphism coming from multiple cover (denoted by k) is already accounted for in $\overline{\mathcal{M}}_{j,j',\beta,\chi}$. We have:

$$[\overline{\mathcal{M}}_{\vec{\Gamma}}/\mathrm{Aut}(\Gamma)] \subset \overline{\mathcal{M}}_{g,n}(X,\beta)^T.$$

Moreover, $[\overline{\mathcal{M}}_{\vec{\Gamma}}/\mathrm{Aut}(\Gamma)]$ is a union of connected components of $\overline{\mathcal{M}}_{g,n}(X,\beta)^T$.

1.4 Toric varieties and toric bundles

In Chapter 2, we would like to move our attention to split toric bundles, a generalization of split projective bundles. Therefore, we would like to review and establish some facts about toric varieties and toric bundles in this section.

1.4.1 Toric varieties

For more details about toric geometry, we refer the readers to [10] and [12]. Let $N = \mathbb{Z}^n$ be a lattice and M its dual. Σ denotes a fan structure on N. For each cone in Σ, we can associate an affine toric variety. Under inclusion of cones, there is an associated open immersions of affine toric varieties. The toric variety X_Σ associated to the fan Σ is the direct limit of affine toric varieties under the direct system of these open immersions. Each affine toric variety corresponding to a maximal dimensional cone is an open chart, and this gives a description of X_Σ in terms of an atlas. Another equivalent construction by GIT quotient will be given later. The rays (one-dimensional cones) are denoted by $\rho_1,\ldots,\rho_{n+k} \in \Sigma[1]$. But by a slight abuse of notation, when using as vectors, we also regard $\rho_i \in N$ as the first lattice point along the ray direction. The torus-invariant divisors are denoted by $D_1,\ldots,D_{n+k}$.

From now on, we assume Σ is complete (any point in N belongs to some cone), and smooth (any cone of dimension n is bounded by n rays whose first lattice points generate N). We have a short exact sequence:

$$0 \to M \to \mathbb{Z}^{n+k} \to A^1(X_\Sigma) \to 0. \tag{1.17}$$

Given $u \in M$, its image in $\mathbb{Z}^{n+k}$ is given by $(u(\rho_1), u(\rho_2), \ldots, u(\rho_{n+k}))$. Given a point $(c_1, c_2, \ldots, c_{n+k}) \in \mathbb{Z}^{n+k}$, its image in $A^1(X_\Sigma)$ is given by $c_1 D_1 + c_2 D_2 + \cdots + c_{n+k} D_{n+k}$. From this description, we can see M parametrizes linear relations among D_i's.

Definition 1.4.1. A *primitive collection* P is a set of distinct rays $\{\rho_{i_1}, \ldots, \rho_{i_l}\}$ such that these rays do not bound a cone, while the rays in any proper subset of P bound a cone.

The cohomology of toric variety is given by the following theorem:

Theorem 1.4.2. *In the $\mathbb{C}[x_1, \ldots, x_{n+k}]$, let $\mathcal{I}$ be the ideal generated by $u(\rho_1)x_1 + u(\rho_2)x_2 + \cdots + u(\rho_{n+k})x_{n+k}$, with u ranging over elements in M. Let $\mathcal{J}$ be the ideal generated by $x_{i_1} \cdots x_{i_l}$, where i_j are distinct and $\rho_{i_1}, \ldots, \rho_{i_l}$ do not bound a cone. We have*

$$H^*(X_\Sigma; \mathbb{C}) = \mathbb{C}[x_1, \ldots, x_{n+k}]/(\mathcal{I} + \mathcal{J}), \tag{1.18}$$

where x_i's are associated with divisors D_i's.

Definition 1.4.3. The ideal $\mathcal{J} \subset \mathbb{C}[x_1, \ldots, x_{n+k}]$ defined above is called *Stanley–Reisner ideal*.

Stanley–Reisner ideal is generated by $D_{i_1} \cdots D_{i_l}$ where $\rho_{i_1}, \ldots, \rho_{i_l}$ form a primitive collection. By relabelling, we can assume $D_1, \ldots, D_k$ are linearly independent in $H^2(X_\Sigma)$, and use them as a basis. Then we can choose a $\mathbb{C}$-basis of $H^*(X_\Sigma)$ consisting of monomials of $D_i, 1 \le i \le k$. Choose a set of multi-indices Λ such that $(D^e)_{e \in \Lambda}$ is a basis (where D^e is the short hand of $D_1^{e_1} \cdots D_k^{e_k}$, $e = (e_1, \ldots, e_k)$). Note that this choice of Λ is not unique but we will fix one from now on.

In (1.17), we can apply $\mathrm{Hom}(-, \mathbb{C}^*)$ to get a short exact sequence of abelian groups

$$1 \to \mathrm{Hom}(A^1(X_\Sigma), \mathbb{C}^*) \to (\mathbb{C}^*)^{n+k} \to \mathrm{Hom}(M, \mathbb{C}^*) \to 1. \tag{1.19}$$

To simplify notations, we write $G = Hom(A^1(X_\Sigma), \mathbb{C}^*)$. $(\mathbb{C}^*)^{n+k}$ acts on $\mathbb{C}^{n+k}$ naturally. By composing with the corresponding map from above, we now have an action of G on

$\mathbb{C}^{n+k}$. Denote the coordinate ring of $\mathbb{C}^{n+k}$ by $\mathbb{C}[z_1, z_2, \ldots, z_{n+k}]$. Given a cone $\sigma \in \Sigma$, Let $\sigma(1)$ be the set of its bounding rays. Let J_Σ be the ideal $(\prod_{\rho_i \notin \sigma(1)} z_i)_{\sigma \in \Sigma}$ and $Z = \mathbb{C}^{n+k} - V(J_\Sigma)$ be the complement open set of the variety given by J_Σ. It is well-known that

Theorem 1.4.4. *The quotient $(\mathbb{C}^{n+k} \backslash Z)/G$ exists as a geometric quotient, and $X_\Sigma \cong (\mathbb{C}^{n+k} \backslash Z)/G$.*

Remark 1.4.5. In fact, by suitably choosing a polarization, Z becomes the underlying set of the unstable locus and X_Σ is the GIT quotient $\mathbb{C}^{n+k} /\!/ G$. The whole story can be written into more details under the framework of GIT, but we only state this simplified version because the GIT presentation is irrelevant here.

Notice $V(J_\Sigma)$ is the union of linear subspaces $\{z_{\rho_i} = 0\}_{\rho_i \in \sigma(1)}$. Now we have a natural action of $(\mathbb{C}^*)^{n+k}$ on X_Σ induced by the quotient map. This action can be described more explicitly. $\mathrm{Hom}(M, \mathbb{C}^*)$ acts naturally on X_Σ by scaling coordinates in each of the affine toric subvarieties (see Chapter 3 in [12]). And we have the map $(\mathbb{C}^*)^{n+k} \to \mathrm{Hom}(M, \mathbb{C}^*)$ from the sequence (1.19). Composing these two, we get a $(\mathbb{C}^*)^{n+k}$ action on X. By a computation of homogeneous coordinates of the toric variety ([10]), this is the same action as before. One can also see the torus-invariant divisor D_i is the image of i-th hyperplanes minus Z under the quotient map.

1.4.2 Toric bundles of split type

All the above construction can be generalized to a relative setting forming toric bundles of split type. Let us review the definition and some basic facts about them. The content should be well-known, but some details are included in order to be self-contained.

Suppose we are given a split vector bundle $V = L_1 \oplus L_2 \oplus \cdots \oplus L_{n+k}$ over a smooth base S. The structure group of the vector bundle is contained in $(\mathbb{C}^*)^{n+k}$. Take the principal bundle $E \to S$ associated to V. Since $(\mathbb{C}^*)^{n+k}$ acts on both E and X_Σ, there is an induced action on $E \times X_\Sigma$.

Definition/Proposition 1.4.6. *The geometric quotient $\mathfrak{X} = (E \times X_\Sigma)/(\mathbb{C}^*)^{n+k}$ exists. It is a fiber bundle over S, and is called a toric bundle (fibration) of split type.*

The existence of $\mathfrak{X}$ is due to an alternative construction by gluing trivializations. Over a point $s \in S$, a transition function between fibers of two trivializations of V can be seen as the action of an element $g(s) \in (\mathbb{C}^*)^{n+k}$. $g(s)$ also acts on the toric variety X_Σ as an automorphism. To construct $\mathfrak{X}$, we simply change the trivializations $U_i \times \mathbb{C}^{n+k} \hookrightarrow V$, $(U_i \subset S)$ into other ones $U_i \times X_\Sigma$. The transition functions of the new fiber bundle are given by the same elements $g(s)$ acting on X_Σ. Locally over S, the quotient map $E \times X_\Sigma \to \mathfrak{X}$ is just induced by the $(\mathbb{C}^*)^{n+k}$ action $(\mathbb{C}^*)^{n+k} \times X_\Sigma \to X_\Sigma$. Obviously this map is $(C^*)^{n+k}$ equivariant and each geometric fiber consists of exactly one orbit.

Correspondingly, we have a quotient description just as Theorem 1.4.4. Since the structure group of V is in $(\mathbb{C}^*)^{n+k}$, we can always trivialize it in a way that coordinates in $\mathbb{C}^{n+k}$ corresponds to L_i directions. There are inclusions $L_1 \oplus L_2 \oplus \cdots \oplus \hat{L}_i \oplus \cdots \oplus L_{n+k} \hookrightarrow V$ given by setting L_i coordinate to 0. Denote the image of these inclusions by $\tilde{D}_i$. These are smooth divisors in $\mathfrak{X}$. Restricting to fibers, $\tilde{D}_i$'s are just the corresponding hyperplanes in $\mathbb{C}^{n+k}$. We still use Z to denote $\bigcap\limits_{\sigma \in \Sigma} \bigcup\limits_{\rho_i \in \sigma(1)} \tilde{D}_i$.

Theorem 1.4.7. *$\mathfrak{X}$ is isomorphic to $(V \backslash Z)/G$.*

The reason is that, locally over S with a given trivialization $U_i \times \mathbb{C}^{n+k} \hookrightarrow V$, the quotient map is just the one in theorem 1.4.4 times the base U_i.

Similarly, we have torus-invariant divisors in $\mathfrak{X}$ with the action of $(\mathbb{C}^*)^{n+k}$ on fibers. By looking at trivializations, these divisors are images of $\tilde{D}_i \backslash (\tilde{D}_i \cap Z)$ under the quotient map. We denote these torus-invariant divisors by $D_1, \ldots, D_{n+k}$. Since a single toric variety is a special case of a toric bundle with $S = \mathrm{spec}(\mathbb{C})$, we override the symbols D_i with the current ones.

Let $\pi : \mathfrak{X} \to S$ be the projection. It induces a map between cohomology rings $\pi^* : H^*(S) \to H^*(\mathfrak{X})$. When there is no confusion, we won't distinguish divisors (divisor classes) and their first Chern classes. By abusing the notation a little bit, we let L_i be $\pi^* c_1(L_i)$.

The cohomology of $\mathfrak{X}$ is described below:

Theorem 1.4.8. *In $H^*(S)[x_1, \ldots, x_{n+k}]$, let $\mathcal{I}$ be the ideal generated by $u(\rho_1)(x_1 - L_1) + u(\rho_2)(x_2 - L_2) + \cdots + u(\rho_{n+k})(x_{n+k} - L_{n+k})$, with u ranging over elements in M. Let $\mathcal{J}$ be the one generated by $x_{i_1} \cdots x_{i_l}$ where $\rho_{i_1}, \ldots, \rho_{i_l}$ are distinct and do not bound a cone. We have the following isomorphism:*

$$H^*(\mathfrak{X}) \cong H^*(S)[x_1, \dots, x_{n+k}]/(\mathcal{I} + \mathcal{J}), \tag{1.20}$$

where the isomorphism sends D_i to x_i.

A reference (maybe not the earliest one in literature) can be the main theorem in [17] which is stated in a more general context.

From now on in writing the presentation of $H^*(\mathfrak{X})$, we will directly use D_i's instead of x_i's.

Example 1.4.9. Consider the case when $X_\Sigma = \mathbb{P}^n$. In this case, $N = \mathbb{Z}^n$. Σ is the only fan structure with rays $e_1, \dots, e_{n+1}$, where $e_1, \dots, e_n$ are unit vectors along each coordinate direction and $e_{n+1} = (-1, \dots, -1) \in N$. V is the split vector bundle over S as before. By looking at a dual basis in M, we have the relations $D_{n+1} - D_i = 0, 1 \leq i \leq n$. Now, we apply the above theorem.

$$\begin{aligned} H^*(\mathfrak{X}) &\cong \frac{H^*(S)[D_1, \dots, D_{n+1}]}{(D_{n+1} - L_{n+1} - D_i + L_i)_{1 \leq i \leq n} + \left(\prod_{j=1}^{n+1} D_j \right)} \\ &\cong \frac{H^*(S)[D_{n+1}]}{\left(\prod_{j=1}^{n+1} (D_{n+1} - L_{n+1} + L_j) \right)} \end{aligned} \tag{1.21}$$

One can check $\mathfrak{X}$ is in fact the projective bundle $\mathbb{P}(V)$. By setting $D_{n+1} - L_{n+1} = h$, this matches with the well-known result of the cohomology of a projective bundle:

$$H^*(\mathfrak{X}) \cong \frac{H^*(S)[h]}{\left(\prod_{j=1}^{n+1} (h + L_j) \right)}. \tag{1.22}$$

Here $\prod_{j=1}^{n+1} (h + L_j)$ is the Chern relation.

1.5 Known results in Gromov–Witten theory of fibrations

This section briefly surveys what is previously known in this direction.

1.5.1 Brown's I-function

As a special family of points on Lagrangian cone (defined in (1.12)), the J-function is defined as

$$J^X(\tau, z^{-1}) = 1 + \frac{\tau}{z} + \sum_{\beta \in \mathrm{NE}(X), n, \mu} \frac{q^\beta}{n!} T_\mu \langle \frac{T^\mu}{z(z-\psi)}, \tau, \dots, \tau \rangle^X_{0,n+1,\beta}, \tag{1.23}$$

where τ is a general element (parametrized by formal parameters) in $H^2(X)$. In other words under the notation of (1.12), $\tau = t(0) = t_0$ and change the variable z into $-z$.

Given a toric bundle of split type $\mathfrak{X}$ over the base S, write $J^S(\tau, z^{-1}) = \sum_{\beta \in N_1(X)} J^S_\beta(\tau, z^{-1}) q^\beta$. For $\beta \notin NE(S)$, we set $J_\beta = 0$. By relabelling, we assume $D_1, \dots, D_k$ (as in section 1.4.2) are $H^*(S)$-linearly independent, and we use them as a basis for the relative Picard group.

In this special case, Let us make a more specific notation about formal variables parametrizing points in $H^2(\mathfrak{X})$. Denote $D = t^1 D_1 + \cdots + t^k D_k$ where t^i are formal variables. Choose a $\mathbb{C}$-basis $(\bar{T}_i)_{1 \le i \le \bar{N}}$ for $H^*(S)$ and denote $\bar{t} = \sum_{i=1}^{\bar{N}} \bar{t}^i \bar{T}_i$ where $\bar{t}^i$ are formal variables. By doing this, D parametrizes a general element in the relative Picard group $Pic(\mathfrak{X}/S)$ and $\bar{t}$ parametrizes the classes $H^2(S)$ from the base.

Brown introduced the I-function of $\mathfrak{X}$

$$I^{\mathfrak{X}}(\bar{t}, D, z, z^{-1}) = e^{D/z} \sum_{\beta \in N_1(\mathfrak{X})} q^\beta e^{(\beta, D)} J^S_{\pi_* \beta}(\bar{t}, z^{-1}) I^{\mathfrak{X}/S}_\beta(z, z^{-1}), \tag{1.24}$$

where $\pi : \mathfrak{X} \to S$ is the projection map from the toric bundle, and

$$I^{\mathfrak{X}/S}_\beta = \frac{\prod_{m=-\infty}^{0} (D_1 + L_1 + mz)}{\prod_{m=-\infty}^{(\beta, D_1 + L_1)} (D_1 + L_1 + mz)} \cdots \frac{\prod_{m=-\infty}^{0} (D_{n+k} + L_{n+k} + mz)}{\prod_{m=-\infty}^{(\beta, D_{n+k} + L_{n+k})} (D_{n+k} + L_{n+k} + mz)}. \tag{1.25}$$

He proved that

Theorem 1.5.1 ([5]). *$I^{\mathfrak{X}}$ lies in the Lagrangian cone of $\mathfrak{X}$.*

This has been an effective way of computing the Gromov–Witten invariant of $\mathfrak{X}$. As a matter of fact, we will prove a quantum Leray–Hirsch and make part of the computation explicit in Chapter 2.

In particular, when $\mathfrak{X}$ is a split projective bundle $\mathbb{P}_S(V)$ with $V = L_1 \oplus \cdots \oplus L_l$, we have

$$I^{\mathbb{P}_S(V)}(\bar{t}, D, z, z^{-1}) = e^{D/z} \sum_{\beta \in N_1(\mathbb{P}_S(V))} q^\beta e^{(\beta, D)} J^S_{\pi_* \beta}(\bar{t}, z^{-1}) \prod_{i=1}^{l} \frac{\prod_{m=-\infty}^{0} (h + L_i + mz)}{\prod_{m=-\infty}^{(\beta, h + L_i)} (h + L_i + mz)}, \tag{1.26}$$

where $h = c_1(\mathcal{O}_{\mathbb{P}_S(V)}(1))$.

It is important to note the following.

Fact. *$I^{\mathbb{P}_S(V)}$ is not symmetric with respect to $c_1(L_i)$. In other words, it is a function in Chern roots of V but not in* Chern classes *of V.*

We suspected (and later confirmed) the Gromov–Witten theory of $\mathbb{P}_S(V)$ only depends on Chern classes of V besides the Gromov–Witten theory of S, though Brown's result doesn't support us directly. This motivates us to develop different techniques. Successful attempts are recorded in Chapter 3 and 4.

CHAPTER 2

QUANTUM LERAY–HIRSCH FOR TORIC FIBRATION

A version of Quantum Leray–Hirsch Theorem is proposed and proven in [20]. A natural idea is to develop it further into split toric fibration as is discussed in this chapter. Essentially, this type of theorems are only reorganizations of the known result by Brown ([5]). But the merit is that the reconstruction process of quantum D-module of a toric fibration is independent of an explicit I-function. Although the PDE system in fiber direction (Picard–Fuchs equations) and the lifting of Dubrovin connections still depend on Brown's I-function, the separation of I-function dependence has been proven useful in applications including the crepant transformation conjecture for ordinary flops in genus-0 ([18, 20]).

In this chapter, we manage to generalize their process to toric bundles of split type. Roughly speaking, the Picard–Fuchs system along the fiber direction can be indexed by primitive collections as in (2.18). It becomes more complicated when writing down the system in the base direction, which includes the admissible liftings introduced in [20] as a key step (See (2.24) and Definition 2.2.4). Finally, we package the system of differential equation in a matrix form as in Theorem 2.2.3. As to the reconstruction of Dubrovin connection, it is described in the section 2.3. We would like to remark that the introduction of primitive collections and the existence of admissible lifting (Theorem 2.2.5) are new.

2.1 Notations

Let $\mathfrak{X}$ be a toric bundle of split type over S. Some of the notations have been adopted in section 1.5, but we would like to systematically set up the notations again:

Convention 3. *1. A bar on the top indicates something related to this basis from the base S. A general cohomology class with formal variables are written as $\bar{t} = \sum_{i=1}^{\bar{N}} \bar{t}^i \bar{T}_i$.*

2. Let $s = c_1\bar{T}_1 + \cdots + c_{\bar{N}}\bar{T}_{\bar{N}} \in H^*(S)$. We slightly abuse notations by letting $\partial_s = c_1\partial_{\bar{t}^1} +$

$\cdots + c_{\bar N}\partial_{\bar t^{\bar N}}$, *as the cohomology basis is hidden in the context.*

3. *If* $\rho_j = \sum_{i=1}^{k} c_i\rho_i$*(no matter whether* $j > k$ *or not), we would still introduce the variable* t^j *and assume the relation* $t^j = \sum_{i=1}^{k} c_i t^i$*. Thus* $\partial_{t^j} = \sum_{i=1}^{k} c_i\partial_{t^i}$*. By introducing these linear relations, we might simplify certain expressions.*

4. *Let* $\partial^{ze} = (z\partial_{t^1})^{e_1}\cdots(z\partial_{t^k})^{e_k}$ *where* t^i*'s are formal variables related with divisors* D_i*. We also write* $\partial^{z(\bar T,e)} = z\partial_s\partial^{ze}$*. By Leray–Hirsch, tensoring an additive basis* $(D^e)_{e\in\Lambda}$ *of* X_Σ *with a basis* (T_i) *of* $H^*(S)$*, we have an additive basis* $(T_i\otimes D^e)_{1\le i\le N, e\in\Lambda}$ *for* $H^*(\mathfrak{X})$*.*

5. *Denote* $\hat t = \bar t + D$ *where* $D = t^1D_1 + \cdots + t^kD_k$*.*

Sometimes we write $J^{\mathfrak{X}}(\tau, z^{-1})$ where a general cohomology class is $\tau = \sum_{i=1}^{N}\tau^iT_i$. A different symbol τ suggests that it might be a function depending on $t^i, \bar t^j$. We write $I^{\mathfrak{X}}(\bar t, D, z, z^{-1})$ to indicate it is a series consisting of positive powers of those variables. Since J-function doesn't involve positive degree terms of z, we write $J^{\mathfrak{X}}(\tau, z^{-1})$. q^β denotes the Novikov variables and we sometimes also include q in variable lists.

Recall in (1.24) the I-function of $\mathfrak{X}$ is

$$I^{\mathfrak{X}}(\bar t, D, z, z^{-1}) = e^{D/z}\sum_{\beta\in N_1(\mathfrak{X})} q^\beta e^{(\beta,D)} J^S_{\pi_*\beta}(\bar t, z^{-1}) I^{\mathfrak{X}/S}_\beta(z, z^{-1}), \tag{2.1}$$

where $\pi : \mathfrak{X}\to S$ is the projection map from the toric bundle, and

$$I^{\mathfrak{X}/S}_\beta = \frac{\prod_{m=-\infty}^{0}(D_1 + L_1 + mz)}{\prod_{m=-\infty}^{(\beta, D_1+L_1)}(D_1+L_1+mz)}\cdots\frac{\prod_{m=-\infty}^{0}(D_{n+k} + L_{n+k} + mz)}{\prod_{m=-\infty}^{(\beta, D_{n+k}+L_{n+k})}(D_{n+k}+L_{n+k}+mz)}. \tag{2.2}$$

There is a function $\tau(\bar t, D)$ and a matrix $B(\tau, z)$ such that

$$(\partial^{z(\bar T_i,e)} I^{\mathfrak{X}}(\bar t, D, z, z^{-1})) = (\nabla J^{\mathfrak{X}}(\tau, z^{-1}))B(\tau, z) \tag{2.3}$$

To write in short,

$$(\partial^{z(\bar T_i,e)} I) = (\nabla J)B. \tag{2.4}$$

This can be proven by basic properties of Lagrangian cones ([20]. For Lagrangian cone formalism, also see [14]). B is invertible and $B(q=0) = id$. Also recall that $\partial_{\tau^i}(\nabla J) = (\nabla J)\bar C_i$ recovers the matrix for quantum multiplication. Most of the effort in this chapter is put into the construction of a similar system of matrix that describes the D-module associated to the I-function:

$$\partial_a(\partial^{z(\bar{T}_i,e)}I) = (\partial^{z(\bar{T}_i,e)}I)C_a(\hat{t},z,q), \tag{2.5}$$

where $a \in \{t^i, \bar{t}^j\}$. Its existence is established in Theorem 2.2.3.

2.2 Quantum Leray–Hirsch for toric bundles of split type

The goal of this section is to look for the matrix $C_a(\hat{t},z,q)$ in (2.5) (Theorem 2.2.3 for a precise statement). To achieve this, we analyze the differential operators (Picard–Fuchs equations) that annihilate the I-function (defined in (1.24)), and reorganize them into the form of (2.5). Let us start with a simple case.

2.2.1 The case when $S = \mathbf{spec}(\mathbb{C})$

Logically, one should head directly to the general case. But it wouldn't hurt to investigate this special case as the whole procedure in this subsection can be applied to the general case in a parallel way to get the Picard–Fuchs system on the fiber direction. In this subsection, we show how primitive collections are used, which is also one of the new ingredients in our chapter comparing to [20].

Given an effective curve $\beta \in NE(\mathfrak{X})$, let $(D_{i_j})_{1\le j\le N_1}$ be the divisors such that $(\beta, D_{i_j}) < 0$. Let $(D_{i'_j})_{1\le j\le N_2}$ be those having $(\beta, D_{i'_j}) > 0$. We introduce the following differential operator

$$\begin{aligned}\Box_\beta = q^\beta e^{(\beta,D)} &\left(\prod_{m=0}^{-(\beta,D_{i_1})-1}(z\partial_{t^{i_1}} - mz)\right)\cdots\left(\prod_{m=0}^{-(\beta,D_{i_{N_1}})-1}(z\partial_{t^{i_{N_1}}} - mz)\right)\\ &- \left(\prod_{m=0}^{(\beta,D_{i'_1})-1}(z\partial_{t^{i'_1}} - mz)\right)\cdots\left(\prod_{m=0}^{(\beta,D_{i'_{N_2}})-1}(z\partial_{t^{i'_{N_2}}} - mz)\right).\end{aligned} \tag{2.6}$$

Proposition 2.2.1. *$\Box_\beta I^{\mathfrak{X}}$ is* 0*.*

Proof. Observe $(z\partial_{t^{i_j}} - mz)I^{\mathfrak{X}}_\beta = (D_{i_1} + (\beta, D_{i_1})z - mz)I^{\mathfrak{X}}_\beta$. The proposition follows from a direct verification. We omit the computations. □

Now, given a monomial basis of cohomology $\{D^e\}_{e\in\Lambda}$, we want to find the associated matrix $\bar{C}_i$ in (2.5). Finding (2.5) amounts to the following question: Given an arbitrary multi-index e', how to express $\partial^{ze'}I^{\mathfrak{X}}$ in terms of a linear combination of $\partial^{ze}I^{\mathfrak{X}} (e \in \Lambda)$ whose coefficients are polynomials in z and power series in q^β, t^i(in fact $q^\beta e^{(\beta,D)}$ as a whole). Because $\Box_\beta$ annihilates $I^{\mathfrak{X}}$, the $\Box_\beta$ gives us some relations among monomial differential operators. It turns out that they give us all we want.

In order to carry out inductions on q, we first of all give a partial order to $NE(\mathfrak{X})$ such that $\beta > \beta'$ if $\beta - \beta' \in NE(X_\Sigma)$.

Consider first the monomial class $D^{e'} \in H^*(\mathfrak{X})$. Using Stanley–Reisner relations, we can turn it into linear combinations of $\{D^e\}_{e\in\Lambda}$. The rough idea of the later arguments is the following. The $\Box_\beta$ consists of two parts: 1. a monic differential operator exactly corresponding to a Stanley–Reisner relation; 2. operators involving nontrivial Novikov variable factors q^β. With this, we can turn $\partial^{ze'}$ into a combination of $\partial^{ze}(e \in \Lambda)$ by a standard q-adic approximation.

Given a primitive collection $P = \{\rho_{i_1}, \dots, \rho_{i_{N_1}}\}$, the sum $\sum_{j=1}^{N_1} \rho_{i_j}$ lies in a certain cone. We write it as $\sum_{j=1}^{N_1} \rho_{i_j} = \sum_{j=1}^{N_2} c_j \rho_{i'_j}$. We can relabel the indices such that $i_1 = i'_1, \dots, i_l = i'_l$, and $i_{l+1}, \dots, i_{N_1} \notin \{\rho_{i'_1}, \dots \rho_{i'_{N_2}}\}$. The primitive collection corresponds to a curve β_P, such that

$$
\begin{aligned}
&(\beta_P, D_{i_j}) = 1 - c_j, && (1 \le j \le l)\\
&(\beta_P, D_{i_j}) = 1, && (j > l)\\
&(\beta_P, D_{i'_j}) = -c_j, && (j > l)\\
&(\beta_P, D_i) = 0, && \text{none of the cases.}
\end{aligned}
\tag{2.7}
$$

According to [10], β_P is effective. And all the extremal curves can be obtained from some primitive collections this way. With this intersection property, we can write $\Box_{\beta_P}$ out explicitly

$$
\begin{aligned}
\Box_{\beta_P} = {} & (z\partial_{t^{i_{l+1}}})(z\partial_{t^{i_{l+2}}}) \cdots (z\partial_{t^{i_{N_1}}}) \\
& - q^{\beta_P} e^{(\beta_P,D)} \prod_{m=0}^{c_1-1} (z\partial_{t^{i'_1}} - mz) \cdots \prod_{m=0}^{c_l-1} (z\partial_{t^{i'_l}} - mz) \prod_{m=0}^{c_{l+1}} (z\partial_{t^{i'_{l+1}}} - mz) \cdots \\
& \cdots \prod_{m=0}^{c_{N_2}} (z\partial_{t^{i'_{N_2}}} - mz).
\end{aligned}
\tag{2.8}
$$

To simplify notations, we write

$$
\Box_{\beta_P} = (z\partial_{t^{i_{l+1}}})(z\partial_{t^{i_{l+2}}}) \cdots (z\partial_{t^{i_{N_1}}}) - q^{\beta_P} e^{(\beta_P,D)} Q_P, \tag{2.9}
$$

where we combine the last a few products into the notation Q_P. The first summand is almost a monic differential operator corresponding to a Stanley–Reisner relation, except that we are missing the factor $(z\partial_{t^{i_1}})(z\partial_{t^{i_2}})\cdots(z\partial_{t^{i_l}})$.

Considering the composition of operators $z\partial_{t^{i_j}}\Box_{\beta_P}$ where $1 \leq j \leq l$, the second summand $q^{\beta_P}e^{(\beta_P,D)}Q_P$ becomes

$$\begin{aligned}
&z\partial_{t^{i_j}}(q^{\beta_P}e^{(\beta_P,D)}Q_P)\\
=&(1-c_j)q^{\beta_P}e^{(\beta_P,D)}Q_P + q^{\beta_P}e^{(\beta_P,D)}z\partial_{t^{i_j}}Q_P\\
=&q^{\beta_P}e^{(\beta_P,D)}((1-c_j)+z\partial_{t^{i_j}})Q_P.
\end{aligned} \tag{2.10}$$

Note it still has a factor $q^{\beta_P}e^{(\beta_P,D)}$. Repeating this process, $(z\partial_{t^{i_1}})(z\partial_{t^{i_2}})\cdots(z\partial_{t^{i_l}})(q^{\beta_P}e^{(\beta_P,D)}Q_P)$ still has a factor $q^{\beta_P}e^{(\beta_P,D)}$. Now we have

$$\begin{aligned}
&(z\partial_{t^{i_1}})(z\partial_{t^{i_2}})\cdots(z\partial_{t^{i_l}})\Box_{\beta_P}\\
=&(z\partial_{t^{i_1}})(z\partial_{t^{i_2}})\cdots(z\partial_{t^{i_{N_1}}}) - q^{\beta_P}e^{(\beta_P,D)}Q'_P,
\end{aligned} \tag{2.11}$$

where Q'_P is a certain operator whose exact form does not concern us. Let us denote $\tilde{\Box}_{\beta_P}$ for $(z\partial_{t^{i_1}})(z\partial_{t^{i_2}})\cdots(z\partial_{t^{i_l}})\Box_{\beta_P}$. In the D-module, this gives us a relation between a Stanley–Reisner type monomial and another operator involving q^{β_P}. To make our goal precise, we want to prove the following.

Theorem 2.2.2. *There is a matrix $C_i(i,q)$ with entries polynomials in z, and power series in q^β's for different effective β's, such that*

$$\partial_{t^i}(\partial^{ze}I) = (\partial^{ze}I)C_i(z,q), \tag{2.12}$$

where $e \in \Lambda$.

Proof. Recall Stanley–Reisner ideal is generated by monomials whose indices form primitive collections. Suppose $P = \{\rho_{i_1},\ldots,\rho_{i_{N_1}}\}$ is a primitive collection. We denote

$$\mathrm{mon}(x,P) = x_{i_1}\cdots x_{i_{N_1}}, \mathrm{mon}(z\partial,P) = z\partial_{t_{i_1}}\cdots z\partial_{t_{i_{N_1}}}. \tag{2.13}$$

Given a multi-index $e' \in \Lambda$,

$$x_j x^{e'} - \sum_{e\in\Lambda} A_e x^e = \sum_P B_P(x)\mathrm{mon}(x,P) + \sum_{u\in M} C_u(x)\sum_j u(\rho_j)x_j, \tag{2.14}$$

where A_e are numbers and B_P, C_u are polynomials. Similarly, we have

$$z\partial_{t^j}\partial^{ze'} - \sum_{e\in\Lambda} A_e \partial^{ze} = \sum_P B_P(z\partial)\mathrm{mon}(z\partial, P), \tag{2.15}$$

where $B_P(z\partial)$ is B_P with x_i replaced by $z\partial_{t^i}$. There is no C_u because $\sum_j u(\rho_j) z\partial_{t^j} = 0$. Therefore, we can write

$$\begin{aligned} &z\partial_{t^j}\partial^{ze'} - \sum_{e\in\Lambda} A_e \partial^{ze} \\ =&\sum_P B_P(z\partial)\tilde{\Box}_{\beta_P} + \sum_P B_P(z\partial)(q^{\beta_P} e^{(\beta_P, D)} Q'_P). \end{aligned} \tag{2.16}$$

Now, recall we have a partial ordering on $NE(\mathfrak{X})$ such that $\beta + \beta' > \beta$ for $\beta, \beta' \in NE(\mathfrak{X}), \beta' \neq 0$. We observe that except the last term, the right-hand side annihilates $I^{\mathfrak{X}}$. Since the last term is a higher order term, we can continue to apply the above operation and invoke an easy induction argument. As a result, $z\partial_{t^j}\partial^{ze'}$ can be written as a combination of $\partial^{ze}, e \in \Lambda$ with coefficient a power series in q. $\square$

What we have proven is a special case of the following theorem in the case when $S = \mathrm{spec}(\mathbb{C})$.

Theorem 2.2.3. *Let $a \in \{t^i, \bar{t}^j\}$ be a formal parameter from either the base or the fibers. There is a matrix $C_a(\hat{t}, z, q)$ whose entries are polynomials in z, but power series in q^β and $\hat{t}$'s for different effective β,$\hat{t}$'s, such that*

$$\partial_a(\partial^{z(\bar{T}_i, e)} I) = (\partial^{z(\bar{T}_i, e)} I) C_a(\hat{t}, z, q). \tag{2.17}$$

Here $e \in \Lambda$ is one of a given multi-indices which give rise to a cohomology basis in the fibers.

2.2.2 The general case

We are going to prove the Theorem 2.2.3 with a general base S.

Given a primitive collection P, we still have a corresponding curve β_P living in some fiber of $\mathfrak{X}$. Recall the fibration is coming from a split vector bundle $V = L_1 \oplus L_2 \oplus \cdots \oplus L_{n+k}$. Now, the box operator becomes

$$\begin{aligned} \Box_{\beta_P} =& (z\partial_{t^{i_{l+1}}} + z\partial_{L_{l+1}}) \cdots (z\partial_{t^{i_{N_1}}} + z\partial_{L_{N_1}}) \\ &- q^{\beta_P} e^{(\beta_P, D)} \prod_{m=0}^{c_1-1} (z\partial_{t^{i'_1}} + z\partial_{L_{i'_1}} - mz) \cdots \prod_{m=0}^{c_l-1} (z\partial_{t^{i'_l}} + z\partial_{L_{i'_l}} - mz) \\ &\prod_{m=0}^{c_{l+1}} (z\partial_{t^{i'_{l+1}}} + z\partial_{L_{i'_{l+1}}} - mz) \cdots \prod_{m=0}^{c_{N_2}} (z\partial_{t^{i'_{N_2}}} + z\partial_{L_{i'_{N_2}}} - mz). \end{aligned} \tag{2.18}$$

We again use Q_P to denote a similar factor as before. Now

$$\square_{\beta_P} = (z\partial_{t^{i_{l+1}}} + z\partial_{L_{l+1}}) \cdots (z\partial_{t^{i_{N_1}}} + z\partial_{L_{N_1}}) - q^{\beta_P} e^{(\beta_P, D)} Q_P. \tag{2.19}$$

As can be seen above, $\square_{\beta_P}$ still relates monomials in ∂_{t^i}, except leaving some higher order derivations in the base direction. Again, a similar computation as before shows that

$$\begin{aligned} \tilde{\square}_{\beta_P} &= (z\partial_{t^{i_1}})(z\partial_{t^{i_2}}) \cdots (z\partial_{t^{i_l}}) \square_{\beta_P} \\ &= (z\partial_{t^{i_1}}) \cdots (z\partial_{t^{i_l}})(z\partial_{t^{i_{l+1}}} + z\partial_{L_{l+1}}) \cdots (z\partial_{t^{i_{N_1}}} + z\partial_{L_{N_1}}) - q^{\beta_P} e^{(\beta_P, D)} Q'_P \end{aligned} \tag{2.20}$$

for some operator Q'_P. We are almost ready to argue by induction in order to rewrite a $z\partial^{ze'}$ into linear combinations of $z\partial^{\bar{T}_i, e}, e \in \Lambda$, except that we haven't understood how to combine $(z\partial_{\bar{t}^i})(z\partial_{\bar{t}^j})$ (from the base cohomology) into a first order differential operator.

Let the quantum differential equation of $H^*(S)$ be given as

$$z\partial_{\bar{t}^i} z\partial_{\bar{t}^j} J^S = \sum_k \bar{C}_{ij}^k(\bar{t}) z\partial_{\bar{t}^k} J^S(\bar{t}). \tag{2.21}$$

Write $\bar{\beta} \in NE(S)$ as an effective curve on the base. The matrix $C_{ij}^k(\bar{t}) = C_{ij,\bar{\beta}}^k(\bar{t}) q^{\bar{\beta}}$. Comparing the $q^{\bar{\beta}}$ term, we have

$$z\partial_{\bar{t}^i} z\partial_{\bar{t}^j} J^S_{\bar{\beta}} = \sum_{k, \bar{\beta}_1} \bar{C}_{ij,\bar{\beta}_1}^k(\bar{t}) z\partial_{\bar{t}^k} J^S_{\bar{\beta} - \bar{\beta}_1}. \tag{2.22}$$

We are going to rewrite $(z\partial_{\bar{t}^i})(z\partial_{\bar{t}^j})$ into certain first order differential operator. Our reduction relies on the following definition extending [20, Definition 3.5]:

Definition 2.2.4. A curve class $\beta \in N_1(X)$ is called *admissible* if $(\beta, D_i) \leq 0$ for all $1 \leq i \leq n+k$. An *admissible lifting* $\bar{\beta}^*$ of $\bar{\beta} \in NE(S)$ is an admissible class such that $\pi_* \bar{\beta}^* = \bar{\beta}$. For admissible $\beta \in N_1(X)$, we define differential operators

$$D_\beta^i = \prod_{m=0}^{-(D_i, \beta)} (z\partial_{t^i} - mz). \tag{2.23}$$

Also let $D_\beta = D_\beta^1 \cdots D_\beta^{n+k}$.

Furthermore, admissible lifting of an effective curve class always exists.

Theorem 2.2.5. *Let $\pi : \mathfrak{X} \to S$ be the projection. For an effective curve $\bar{\beta} \in NE(S)$, there is a curve $\beta \in NE(\mathfrak{X})$ such that $\pi_* \beta = \bar{\beta}$, and $(D_i, \beta) \leq 0$ for all $1 \leq i \leq n+k$.*

Let us first of all look at the consequence of the theorem. To simplify notations, we write $\bar{\beta} = \pi_*\beta$, in the following reduction. Also write $\bar{t} = \bar{t}_1 + \bar{t}_2$ with $\bar{t}_1$ the divisor part. $\bar{\beta}^*$ is an arbitrary choice of an admissible lifting of $\bar{\beta}$. Apply $(z\partial_{\bar{t}^i})(z\partial_{\bar{t}^j})$ to (1.24), we have

$$\begin{aligned}
&z\partial_{\bar{t}^i} z\partial_{\bar{t}^j} I^{\mathfrak{X}} \\
&= e^{D/z} \sum_{\beta\in N_1(\mathfrak{X})} q^\beta e^{(\beta,D)} I_\beta^{\mathfrak{X}/S} z\partial_{\bar{t}^i} z\partial_{\bar{t}^j} J^S_{\bar{\beta}} \\
&= e^{D/z} \sum_{\substack{\beta\in N_1(\mathfrak{X}) \\ \bar{\beta}_1\in NE(S),k}} q^\beta e^{(\beta,D)} I_\beta^{\mathfrak{X}/S} \bar{C}^k_{ij,\bar{\beta}_1} z\partial_{\bar{t}^k} J^S_{\bar{\beta}-\bar{\beta}_1} \\
&= \sum_{\bar{\beta}_1,k} q^{\bar{\beta}_1^*} e^{(D,\bar{\beta}_1^*)} C^k_{ij,\bar{\beta}_1} z\partial_{\bar{t}^k} \sum_\beta q^{\beta-\bar{\beta}_1^*} e^{\frac{D}{z}+(\beta-\bar{\beta}_1^*,D)} I_\beta^{\mathfrak{X}/S} J^S_{\bar{\beta}-\bar{\beta}_1}(\bar{t}) \\
&= \sum_{\bar{\beta}_1,k} q^{\bar{\beta}_1^*} e^{(D,\bar{\beta}_1^*)} C^k_{ij,\bar{\beta}_1} z\partial_{\bar{t}^k} \sum_\beta D_{\bar{\beta}_1^*} q^{\beta-\bar{\beta}_1^*} e^{\frac{D+\bar{t}_1}{z}+(\beta-\bar{\beta}_1^*,D+\bar{t}_1)} I_\beta^{\mathfrak{X}/S} J^S_{\bar{\beta}-\bar{\beta}_1}(\bar{t}_2) \\
&= \sum_{\bar{\beta}_1,k} q^{\bar{\beta}_1^*} e^{(D,\bar{\beta}_1^*)} C^k_{ij,\bar{\beta}_1} z\partial_{\bar{t}^k} D_{\bar{\beta}_1^*} I^{\mathfrak{X}}.
\end{aligned} \tag{2.24}$$

The constant term looks like:

$$z\partial_{\bar{t}^i} z\partial_{\bar{t}^j} I^{\mathfrak{X}} \equiv \sum_k C^k_{ij,0} z\partial_{\bar{t}^k} I^{\mathfrak{X}} \qquad (\text{mod } q). \tag{2.25}$$

As a result, by a straightforward induction on q, we can say derivatives in the base directions of any order can be reduced into a sum of operators of the form $z\partial_{\bar{t}_k}\partial^{ze'}$ with coefficients power series in q.

Let us write $|e| = \sum_{i=1}^{n+k} e_i$. To prove theorem (2.2.3), we begin with a multi-index $e' \in \Lambda$. As before, we have

$$x_j x^{e'} - \sum_{e\in\Lambda} A_e x^e = \sum_P B_P(x)\mathrm{mon}(x,P) + \sum_{u\in M} C_u(x) \sum_j u(\rho_j)(x_j - L_j). \tag{2.26}$$

By comparing degrees, we can assume $A_e \neq 0$ whenever $|e| < |e'|$. We can write

$$\begin{aligned}
&z\partial_{t^j}\partial^{ze'} - \sum_{e\in\Lambda} A_e \partial^{ze} \\
&= \sum_P B_P(z\partial)\tilde{\Box}_{\beta_P} + \sum_P B_P(z\partial)(q^{\beta_P} e^{(\beta_P,D)} Q'_P) - \\
&\sum_P \Big((z\partial_{L_{l+1}}(z\partial_{t^{i_1}} \cdots \widehat{z\partial_{t^{i_l-1}}} \cdots z\partial_{t^{i_{N_1}}}) + \cdots) + \\
&(z\partial_{L_{l+1}} z\partial_{L_{l+2}}(z\partial_{t^{i_1}} \cdots \widehat{z\partial_{t^{i_{l+1}}}} \cdots z\partial_{t^{i_{l+2}}} z\partial_{t^{i_{N_1}}}) + \cdots) + \cdots \Big).
\end{aligned} \tag{2.27}$$

The last line comes from the expansion of

$$(z\partial_{t^{i_1}}) \cdots (z\partial_{t^{i_l}})(z\partial_{t^{i_{l+1}}} + z\partial_{L_{l+1}}) \cdots (z\partial_{t^{i_{N_1}}} + z\partial_{L_{N-1}}) \tag{2.28}$$

in $\tilde{\Box}_{\beta_P}$. Every term in the last two lines consists of some derivatives in the base direction, and some in the fiber direction. The fiber-direction terms, e.g., $(z\partial_{t^{i_1}} \cdots \widehat{z\partial_{t^{i_l-1}}} \cdots z\partial_{t^{i_{N_1}}})$ all have order less than $|e'|$. We can impose an induction hypothesis on $|e'|$ and assume derivatives in the fiber directions of order less than e' are reduced to a first order one in the fiber direction and a higher order one in the base direction. But as we discussed in the admissible lifting, higher order derivatives in the base direction can be reduced to a sum of first order ones, plus terms with higher q-degree. Cooperating with q-adic approximation, $z\partial_{t^j}\partial^{ze'}$ can be written into a sum of derivatives $z\partial_{\bar{t}^k}\partial^{ze}, (e \in \Lambda)$ with coefficients power series in q. One argues similarly for monomials of the form $z\partial_{\bar{t}_j}z\partial_{\bar{t}_k}\partial^{ze'}$. Thus, Theorem (2.2.3) is proven.

Proof of Proposition 2.2.5. Given a maximal cone $\sigma \in \Sigma$, let $\rho_{i_1}, \ldots, \rho_{i_n}$ be the extremal rays of σ. Regarding D_i as cycles in $A_{n-1}(\mathfrak{X})$, we observe that $D_{i_1} \cap \cdots \cap D_{i_n}$ restricts to a torus-invariant point on every fiber. By looking at trivializations, this can be realized as the image of a section $s_\sigma : S \to \mathfrak{X}$. In other words, $D_{i_1} \cap \cdots \cap D_{i_n} = (s_\sigma)_*[S]$. With a given $\bar{\beta} \in NE(S)$, $(s_\sigma)_*\bar{\beta}$ is a lifting of $\bar{\beta}$, as $\pi_*(s_\sigma)_*\bar{\beta} = \bar{\beta}$. Also, $(s_\sigma)_*\bar{\beta} = (s_\sigma)_*((s_\sigma)^*\pi^*\bar{\beta} \cap [S]) = \pi^*\bar{\beta} \cap (s_\sigma)_*[S] = \pi^*\bar{\beta} \cap D_{i_1} \cap \cdots \cap D_{i_n}$, where $(s_\sigma)^*$ is the Gysin morphism whose compatibility with flat pull-back is justified in [13].

Now,

$$(s_\sigma)_*\bar{\beta} \cap D_j = \pi^*\bar{\beta} \cap D_{i_1} \cap \cdots \cap D_{i_n} \cap D_j. \tag{2.29}$$

Notice if $j \notin \{i_1, \ldots, i_n\}$, this product is 0, since $\rho_{i_1}, \ldots, \rho_{i_n}, \rho_j$ do not bound a cone. We are left with $(s_\sigma)_*\bar{\beta} \cap D_{i_l}$ for some $1 \le l \le n$. Consider the dual basis $\rho_{i_1}^\vee, \ldots, \rho_{i_n}^\vee \in M$ of $\rho_{i_1}, \ldots, \rho_{i_n}$. Its existence is guaranteed by the smoothness assumption on Σ. Apply $\rho_{i_l}^\vee$ to the rays, we get a relation among divisors:

$$D_{i_l} - L_{i_l} = -\sum_{j \notin \{i_1, \ldots, i_n\}}^{n+k} \rho_{i_l}^\vee(\rho_j)(D_j - L_j). \tag{2.30}$$

We substitute D_{i_l} by a combination of above D_j's and L_j's. Summands like D_j vanish in the product, since $j \notin \{i_1, \ldots, i_n\}$. And summands like L_j are pull-backs from the base. Our first goal is to compute the intersection number $((s_\sigma)_*\bar{\beta}, D_{i_l})$. Numerically (in $N_*(\mathfrak{X})$), we have

$$
\begin{aligned}
&(s_\sigma)_*\bar{\beta} \cap D_{i_l} \\
&= \pi^*\bar{\beta} \cap D_{i_1} \cap \cdots \cap D_{i_n} \cap D_{i_l} \\
&= \pi^*\bar{\beta} \cap D_{i_1} \cap \cdots \cap D_{i_n} \cap (L_{i_l} - \sum_{j \notin \{i_1,\ldots,i_n\}}^{n+k} \rho_{i_l}^{\vee}(\rho_j)(D_j - L_j)) \\
&= \pi^*\bar{\beta} \cap D_{i_1} \cap \cdots \cap D_{i_n} \cap (L_{i_l} + \sum_{j \notin \{i_1,\ldots,i_n\}}^{n+k} \rho_{i_l}^{\vee}(\rho_j)L_j) \qquad (2.31) \\
&= \pi^*\bar{\beta} \cap D_{i_1} \cap \cdots \cap D_{i_n} \cap (\sum_{j=1}^{n+k} \rho_{i_l}^{\vee}(\rho_j)L_j) \\
&= (s_\sigma)_*(\bar{\beta} \cap (\sum_{j=1}^{n+k} \rho_{i_l}^{\vee}(\rho_j)L_j)) \\
&= (\bar{\beta}, \sum_{j=1}^{n+k} \rho_{i_l}^{\vee}(\rho_j)L_j).
\end{aligned}
$$

The last equality is because we push forward a 0-cycle, and take its numerical class.

Denote $n_i = (\bar{\beta}, L_i)$ and let us let the maximal cone σ vary. We are going to find a cone such that the above number is nonpositive. Now the original problem can be rephrased purely in terms of fan structure.

Lemma 2.2.6. *Σ is a smooth fan. Given arbitrary integer numbers n_i, there is a cone $\sigma \in \Sigma$ of dimension n, with extremal rays $\rho_{i_1}, \ldots, \rho_{i_n}$, such that*

$$\sum_{j=1}^{n+k} \rho_{i_l}^{\vee}(\rho_j)n_j \leq 0 \qquad (2.32)$$

for all $1 \leq l \leq n$.

Proof. First we construct a linear functional from n_i's. Given a $u \in M$, set

$$f(u) = \sum_{i=1}^{n+k} u(\rho_i)n_i \qquad (2.33)$$

Because $M^\vee = N$, f can be naturally viewed as an element in N. Now, due to the completeness of the fan Σ, there exists a maximal cone σ such that $-f \in \bar{\sigma}$, the closure of the cone. We will show that this is the cone that we want. Since $-f$ falls in the closure of the cone, it can be expressed as a sum of extremal rays with nonnegative coefficients. Say $-f = \sum_{j=1}^{n} c_j \rho_{i_j}$ with $c_j > 0$. Pick a $u \in M$ and take the pairing with it to both sides, we have

$$\sum_{j=1}^{n} c_j u(\rho_{i_j}) = -f(u) = -\sum_{i=1}^{n+k} u(\rho_i) n_i. \tag{2.34}$$

The equality to the right is from (2.33). As a result,

$$\begin{aligned}(s_\sigma)_* \bar{\beta} \cap D_{i_l} &= (\bar{\beta}, \sum_{j=1}^{n+k} \rho_{i_l}^\vee(\rho_j) L_j) \\ &= \sum_{j=1}^{n+k} \rho_{i_l}^\vee(\rho_j) n_j = -\sum_{j=1}^{n} c_j \rho_{i_l}^\vee(\rho_{i_j}) \leq 0. \end{aligned} \tag{2.35}$$

The second to the last equality follows by taking u to be $\rho_{i_l}^\vee$, and the last inequality follows from the property $c_j > 0$ and $\rho_{i_l}^\vee(\rho_{i_j})$ is either 1 or 0. □

Now that the lemma is established, $(s_\sigma)_* \bar{\beta}$ is an admissible lifting. Thus, Theorem 2.2.5 follows. □

2.3 Birkhoff factorization

In this section, we try to understand the Birkhoff factorization and the structure matrix of quantum D-module following the procedure in [20]. Let us suppress the variable q in those formal computation below. First by Theorem 2.2.3, we have

$$z\partial_a(\partial^{z(\bar{T}_i, e)} I(\hat{t}, z, z^{-1})) = (\partial^{z(\bar{T}_i, e)} I(\hat{t}, z, z^{-1})) C_a(\hat{t}, z). \tag{2.36}$$

In this section, we compute the Birkhoff factorization $B(z, z^{-1}, \tau(\hat{t}))$ inductively along with the structural matrix of the quantum ring composed with mirror transformation $\tilde{C}_\mu(\tau(\hat{t}))$. Here $\hat{t} = \bar{t} + D$, $a \in \{t^i, \bar{t}^j\}$, and $B, \tilde{C}_a$ satisfy:

$$(\partial^{z(\bar{T}_i, e)} I(\hat{t}, z, z^{-1})) = (\nabla J(\tau, z^{-1})) B(\tau, z) \tag{2.37}$$

and

$$z\partial_\mu(\nabla J(\tau, z^{-1})) = (\nabla J(\tau, z^{-1})) \tilde{C}_\mu(\tau). \tag{2.38}$$

Note that $\tau(\hat{t})$ is the (generalized) mirror transformation in (2.3), and μ is the index to indicate the difference of variables, i.e., $\tau = \sum \tau^\mu T_\mu$, and $\partial_\mu = \dfrac{\partial}{\partial \tau^\mu}$. In general, the mirror transformation obtained from Brown's I-function $\tau(\hat{t})$ may not be invertible. Therefore, the matrix $\tilde{C}_\mu(\tau)$ does not immediately recover the structure of the quantum ring of $\mathfrak{X}$.

Also recall that both $(\partial^{z(\bar{T}_i,e)} I(\hat{t},z,z^{-1}))$ and $(\nabla J(\tau,z^{-1}))$ are matrices whose column vectors lined up according to the chosen basis $T_i \cup D^e$.

We suppress z, z^{-1} variables for now. Substituting (2.36) by (2.37), we get

$$z\partial_a((\nabla J(\tau))B(\tau)) = (\nabla J(\tau))B(\tau)C_a(\hat{t}). \tag{2.39}$$

Thus we have

$$z\partial_a(\nabla J(\tau)) = (\nabla J(\tau))[B(\tau)C_a(\hat{t})B^{-1}(\tau) - z\partial_a B(\tau)B^{-1}(\tau)] \tag{2.40}$$

Comparing (2.38), we need to take the mirror transformation $\tau(\hat{t})$ into account:

$$\begin{aligned} z\partial_{t^a}(\nabla J(\tau(\hat{t}))) &= \sum_\mu \frac{\partial \tau^\mu(\hat{t})}{\partial a} z\partial_\mu(\nabla J(\tau(\hat{t}))) \\ &= (\nabla J(\tau(\hat{t}))) \sum_\mu \tilde{C}_\mu(\tau(\hat{t})) \frac{\partial \tau^\mu(\hat{t})}{\partial a}. \end{aligned} \tag{2.41}$$

To simplify, we shall denote $\tilde{C}'_a(\hat{t}) = \sum_\mu \tilde{C}_\mu(\tau(\hat{t})) \frac{\partial \tau^\mu(\hat{t})}{\partial a}$. Comparing this with (2.40), we have

$$\tilde{C}'_a(\hat{t}) = B(\tau)C_a(\hat{t})B^{-1}(\tau) - z\partial_a B(\tau)B^{-1}(\tau). \tag{2.42}$$

What we shall get eventually are $B(\tau(\hat{t}))$ and the $\tilde{C}'_a(\hat{t})$. Now let us remember the z, z^{-1} variables in the above equation:

$$\tilde{C}'_a(\hat{t}) = B(\tau,z)C_a(\hat{t},z)B^{-1}(\tau,z) - z\partial_a B(\tau,z)B^{-1}(\tau,z). \tag{2.43}$$

Notice the left-hand side is independent of z. Thus, by separating degrees in z, we have the following two equations:

$$\tilde{C}'_a(\hat{t}) = B_0(\tau)C_{a,0}(\hat{t})B_0^{-1}(\tau) \tag{2.44}$$

and

$$z\partial_a B(\tau,z) = B(\tau,z)C_a(\hat{t},z) - B_0(\tau)C_{a,0}(\hat{t})B_0^{-1}(\tau)B(\tau,z), \tag{2.45}$$

where $B_0, C_{a,0}$ mean the constant terms as polynomials of z. B_0^{-1} means $(B^{-1})_0$ the constant term of the inverse matrix. (2.45) is obtained by multiplying B to both sides, comparing nonconstant terms, and substitute $\tilde{C}'_a$ according to (2.44).

Now, we can make an inductive algorithm according to (2.45). Let us keep in mind that B is always composed with $\tau(\hat{t})$, and omit the τ variable in our computation. As before, we give $NE(\mathfrak{X})$ a partial order such that $\beta+\beta' > \beta$, with $\beta, \beta' \in NE(\mathfrak{X})$. Then we separate coefficients of q^β:

$$\begin{aligned}&z\partial_a[B(z)]_\beta\\ &= \sum_{\beta_1+\beta_2=\beta}[B(z)]_{\beta_1}[C_a(z)]_{\beta_2} - \sum_{\beta_1+\beta_2+\beta_3+\beta_4=\beta}[B_0]_{\beta_1}[C_{a,0}]_{\beta_2}[B_0^{-1}]_{\beta_3}[B(z)]_{\beta_4},\end{aligned} \tag{2.46}$$

where $[\cdot]_\beta$ means the coefficients of q^β.

First of all, $[B(z)]_0 = id$. Most of the terms in the right-hand side involve data $[B]_{\beta'}$ with $\beta' < \beta$ except the following:

$$[B(z)]_\beta[C_a(z)]_0 - [C_{a,0}]_0[B(z)]_\beta - [B_0]_\beta[C_{a,0}]_0 - [C_{a,0}]_0[B_0^{-1}]_\beta. \tag{2.47}$$

But we have the following property of C_a:

Lemma 2.3.1. *We have the equality* $[C_a(z)]_0 = [C_{a,0}]_0$.

Proof. It basically says that in the matrix C_a, the constant term (in q) does not involve any z. We know that $I \cong e^{D/z}(\text{mod q})$, and $z\partial_{t^i}I \cong D_i e^{D/z}(\text{mod q})$. So we just compare the q constant parts in (2.36). A straightforward verification shows that the constant terms of $z\partial_a\partial^{(z,e)}I, \partial^{(z,e)}I$ do not involve z. So the same should happen for C_a. □

This suggests that the degree of z in the term above is strictly less than the degree of z in $z\partial_a[B(z)]_\beta$. Thus, it suggests that the degree d (in variable z) part of $[B(z)]_\beta$ is determined by $[B(z)]_{\beta'}$ with $\beta' < \beta$, and the degree $d-1$ part of $[B(z)]_\beta$. An induction on both β and the degree of z reconstructs $B(z)$ for us. Once we have the $B(z)$, we can solve the $\tilde{C}'_a(\hat{t})$ by (2.44).

Now that $\tilde{C}'_a(\hat{t})$ is obtained, we can finally recover the mirror transformation $\tau(\hat{t})$. Recall $\tilde{C}'_a(\hat{t}) = \sum_\mu \tilde{C}_\mu(\tau(\hat{t}))\frac{\partial\tau^\mu(\hat{t})}{\partial a}$. Plugging in the explicit formula for $\tilde{C}_\mu$ and write down the expression for a specific entry in the matrix, we get

$$[\tilde{C}'_a(\hat{t})]_i^j = \sum_\mu \frac{q^\beta}{n!}\langle T_\mu, T_i, T^j, \tau(\hat{t})^{\circ n}\rangle_{0,n,\beta}\frac{\partial\tau^\mu(\hat{t})}{\partial t^a}. \tag{2.48}$$

We can look at a specific column

$$\begin{aligned}
[\tilde{C}'_a(\hat{t})]^j_0 &= \sum_\mu \frac{q^\beta}{n!}\langle T_\mu, 1, T^j, \tau(\hat{t})^{\circ n}\rangle_{0,n,\beta} \frac{\partial \tau^\mu(\hat{t})}{\partial t^a} \\
&= \sum_\mu \langle T_\mu, 1, T^j\rangle_{0,3,0} \frac{\partial \tau^\mu(\hat{t})}{\partial t^a} \\
&= \frac{\partial \tau^j(\hat{t})}{\partial t^a}.
\end{aligned} \tag{2.49}$$

So we recover all partial derivatives of $\tau(\hat{t})$. Using the fact $\tau(\hat{t}) = \hat{t}$(mod q), we recover $\tau(\hat{t})$.

Remark 2.3.2. With $\tau(\hat{t}), B(\tau, z), \tilde{C}'_a(\hat{t})$, we only understand the quantum D-module after a change of variable. But since $\tau(\hat{t})$ is in general non-invertible, we don't recover the full quantum D-module immediately. However, the reconstruction of the full quantum D-module is still possible with the help of the divisorial reconstruction formula in [23, Theorem 1]. One can use this to reduce to insertions of the form $T_i \cup D^e$ into insertions T_i from the base and divisors D_j. The details are outside the scope of the chapter.

CHAPTER 3

PROJECTIVE BUNDLES OVER GKM MANIFOLD VIA RECURSION

Along with the study of split toric fibrations, we also want to explore the Gromov–Witten theory of "non-split toric fibrations" (which doesn't seem to have a generally accepted definition as far as we are aware of). In particular, we are interested in non-split projective bundles, since the notion of "projective bundle" is classically defined and they naturally appear in different places. Recall in section 1.1 Question 1, we want to understand whether the Gromov–Witten theory of a projective bundle is determined by that of the base and the Chern classes. More precisely, one can ask whether $\mathbb{P}_Y(V_1)$ and $\mathbb{P}_Y(V_2)$ have the same Gromov–Witten theory whenever $c(V_1) = c(V_2)$. To answer this question, we would like to consider the special case when the base Y is a GKM manifold under a torus action, and the vector bundle V is equivariant. Under these conditions, we get an affirmative answer to the above question whose rigorous statement will be spelled out in the next section.

3.1 Statement of the result

Let Y be a smooth projective variety with a T-action and V be a T-equivariant, *not necessarily split*, vector bundle of rank r over Y. The projective bundle $\mathbb{P}(V)$ naturally carries an induced T-action. The equivariant cohomology of $\mathbb{P}(V)$ has the following presentation

$$H_T^*(\mathbb{P}(V)) = \frac{H_T^*(Y)[h]}{(c_T(V)(h))}, \tag{3.1}$$

where $h = c_1(\mathcal{O}(1))$ and $c_T(V)(x) = \sum x^{r-i} c_i(V)$. Given two such equivariant vector bundles V_1, V_2 over Y with the same equivariant Chern classes $c_T(V_1) = c_T(V_2)$, the equivariant cohomology/Chow are canonically isomorphic by the above presentation

$$\mathfrak{F} : H_T^*(\mathbb{P}(V_1)) \cong H_T^*(\mathbb{P}(V_2)). \tag{3.2}$$

$\mathfrak{F}$ induces an isomorphism between $N_1(\mathbb{P}(V_1))$ and $N_1(\mathbb{P}(V_2))$ by the intersection pairing, and we slightly abuse the notation and denote the induced isomorphism also by $\mathfrak{F}$. This isomorphism on curve classes is uniquely characterized by the property: $(\mathfrak{F}D, \mathfrak{F}\beta) = (D, \beta)$ for any $D \in N^1(\mathbb{P}(V_1)), \beta \in N_1(\mathbb{P}(V_1))$.

Theorem A (=Theorem 3.3.2)**.** *If Y is a projective smooth variety with a torus action such that there are finitely many fixed points and one-dimensional orbits, then the $\mathfrak{F}$ induces an isomorphism of T-equivariant genus 0 Gromov–Witten invariants between $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$*

$$\langle \psi^{a_1}\sigma_1, \ldots, \psi^{a_n}\sigma_n \rangle_{0,n,\beta}^{\mathbb{P}(V_1)} = \langle \psi^{a_1}\mathfrak{F}\sigma_1, \ldots, \psi^{a_n}\mathfrak{F}\sigma_n \rangle_{0,n,\mathfrak{F}\beta}^{\mathbb{P}(V_2)}. \tag{3.3}$$

Remark 3.1.1. Such Y are often called a proper algebraic GKM manifold and examples include toric varieties, Grassmannians, flag varieties, and certain Hilbert schemes of points.

3.2 A recursion relation

The preliminary of virtual localization is covered in section 1.2.4. In this section, we state and prove a technical result (Theorem 3.2.4), generalizing A. Givental's theorem in [5, Theorem 2]. It is a way of packaging the virtual localization into a recursive form and will be applied in section 3.3 to prove our main result Theorem 3.3.2.

3.2.1 Statement

Before stating the theorem, a few definitions are in order. Recall that, given a meromorphic function $f(z)$, the *principal part* at $z = a$, denoted $\mathbf{Prin}_{z=a} f$, is a polynomial in $1/(z-a)$ without a constant term such that $f - \mathbf{Prin}_{z=a} f$ has no pole at $z = a$.

Let F be a family of points in the Lagrangian cone introduced earlier and let $F^j := \iota_j^* F$. Localization formula implies that

$$F = \sum_{j=1}^{l} (\iota_j)_! \frac{F^j}{e_T(N_j)}. \tag{3.4}$$

When we restrict via ι_j, the following function spaces and polarizations are used.

Convention 4. *Let $\mathcal{H}^j$ be*

$$\mathcal{H}^j := H^*(Z_j; S_T) \left[z, \frac{1}{z}\right] [\mathfrak{S}] [\![Q]\!], \tag{3.5}$$

where $\mathfrak{S} = \left\{ \dfrac{1}{z+\chi} \right\}_{\chi \in C(T)_{\mathbb{Q}}}$. *The polarization is given by* $\mathcal{H}^j = \mathcal{H}^j_+ \oplus \mathcal{H}^j_-$ *such that* $\mathcal{H}^j_+$ *has no* z^{-1} *(but has* $(z+\chi)^{-1}$*) and*

$$\mathcal{H}^j_- = z^{-1} H^*(Z_j; S_T)[z^{-1}][\![Q]\!].$$

The Lagrangian cone in the function space $\mathcal{H}^j$ *under the above polarization is denoted by* $\mathcal{L}^j_E$.

Remark 3.2.1. Even though the above definition of $\mathcal{H}^j$ appears to involve polynomials of infinite indeterminates, in practice this is not needed. Given a torus action on a fixed X, there are only finitely many integral characters $\chi \in C(T)$ associated to all one-dimensional orbits in X. For a fixed d in Q^d, the *fractional characters* involved are of the form χ/k, where χ are the integral characters of one-dimensional orbits, and k is linearly bounded by the numerical class d.

A priori, we do not have $F^j \in (\mathcal{H}^j, -1z)$, but the localization computation in the next subsection allows us to write F^j as an element in the formal neighborhood $(\mathcal{H}^j, -1z)$. F as a Laurent series in z^{-1} can be recovered using localization formula (3.4) by expanding $(z+\chi)^{-1}$ as power series in z^{-1}. We note that F^j satisfies the following properties.

- The Novikov variables Q^d still have exponents $d \in \mathrm{NE}(X)$.
- The coefficient of Q^d for a fixed d is a polynomial instead of a power series in z^{-1}.

Remark 3.2.2. The polarization $\mathcal{H}^j = \mathcal{H}^j_+ \oplus \mathcal{H}^j_-$ is a form of partial fraction decomposition. That is, any element $p(z) \in \mathcal{H}^j$ can be uniquely decomposed into

$$p(z) = [p(z)]_+ + [p(z)]_\bullet + [p(z)]_-, \tag{3.6}$$

where

$$\begin{aligned}
&[p(z)]_+ \in H^*(Z_j; S_T)[z][\![Q]\!], \\
&[p(z)]_\bullet \in \bigoplus_{\chi \in C(T)_{\mathbb{Q}}} (z+\chi)^{-1} H^*(Z_j; S_T)[(z+\chi)^{-1}][\![Q]\!], \text{ and} \\
&[p(z)]_- \in z^{-1} H^*(Z_j; S_T)[z^{-1}][\![Q]\!].
\end{aligned} \tag{3.7}$$

This also induces a decomposition of elements in $(\mathcal{H}^j, -1z)$.

Remark 3.2.3. We use notation $\mathcal{L}_E^j$ other than $\mathcal{L}_{Z_j,E}$ to emphasize the different underlying function spaces and polarizations. Under this new polarization, the points in $\mathcal{L}_E^j$ allow $(z+\chi)^{-1}$ terms in $t(z)$, i.e.,

$$t(z) \in H^*(Z_j; S_T)[\![t_i^1, \ldots, t_i^N]\!][z, (z+\chi)^{-1}][\![Q]\!],\ i \geq 0,\ \chi \in C(T)_{\mathbb{Q}}, \tag{3.8}$$

where $N = \dim(H^*(Z_j; \mathbb{C}))$.

We now state the recursion relation.

Theorem 3.2.4. *Assume all $e_T(E_{0,n,d})^{-1}$ involved are defined. There exist equivariant cohomology classes $A_{i,k;j,j',\beta,\chi} \in H_T^*(\overline{\mathcal{M}}_{j,j',\beta,\chi}) \otimes_{R_T} S_T$ such that (a formal family of point) $F \in (\mathcal{H}, -1z)$ lies in $\mathcal{L}_{X,E}$ if and only if its restrictions F^j can be written as elements in $(\mathcal{H}^j, -1z)$ and satisfy the following conditions:*

(1) $F^j(-z) \in \mathcal{L}^j_{N_j \oplus E}$ (under the indicated polarization);

(2) The principal parts of $\{F^j(-z)\}$ satisfy recursively defined expressions:

$$\begin{aligned}&\mathop{\mathbf{Prin}}_{z=-\chi} F^j(z)\\ &= \sum_{i,k,j',d,\chi} (ev_+)_*^{vir} \left[\frac{Q^d A_{i,k;j,j',\beta,\chi} \dfrac{\partial^k}{\partial w^k} \left[(ev_-)^* \left(F^{j'}(w) - \mathop{\mathbf{Prin}}_{w=-\chi} F^{j'}(w) \right) \right] \Big|_{w=-\chi}}{(z+\chi)^i} \right],\end{aligned} \tag{3.9}$$

where ev_- is the map defined on $\overline{\mathcal{M}}_{j,j',\beta,\chi}$.

Furthermore, $A_{i,k;j,j',\beta,\chi} \in H_T^*(\overline{\mathcal{M}}_{j,j',\beta,\chi})$ are uniquely determined by

(i) the classes of equivariant vector bundles $[(TX \oplus E)|_{X_{j,j',\beta,\chi}}] \in K_T^0(X_{j,j',\beta,\chi})$,

(ii) the classes of equivariant vector bundles $[N_j \oplus E] \in K_T^0(Z_j)$,

(iii) the moduli stacks $\overline{\mathcal{M}}_{j,j',\beta,\chi}$, their virtual classes, and their universal families $\mathcal{C}_{j,j',\beta,\chi}$.

Condition (2) is sometimes referred to as the *"recursion relation"*. The recursion relation comes from virtual localization, cf., (3.18) in the proof below.

Remark 3.2.5. The above sum for $\mathop{\mathbf{Prin}}_{z=-\chi} F^j(z)$ is finite in i, k, j', χ for a fixed d when the target variety is of finite type. That is, for a fixed d, all but finitely many $A_{i,k;j,j',\beta,\chi}$ vanish. This follows from the proof given below.

We state a corollary to be used in the last section. Let X be a smooth variety with a torus actions by T. Let W be a smooth invariant subvariety of X. Assume $e_T((N_{W/X})_{0,n,d})^{-1}$ are defined (or defined as a limit according to Remark 1.2.4). We compare the recursion relations.

Corollary 3.2.6. *X, W as above. Fix a $\chi \in C(T)_{\mathbb{Q}}$. Fix a $Z_j \subset W$. Suppose for any $1 \le j' \le l$, $d \in \mathrm{NE}(X)$, we have either $\overline{\mathcal{M}}_{j,j',\beta,\chi} = \emptyset$ or $X_{j,j',\beta,\chi} \subset W$. Then $A_{i,k;j,j',\beta,\chi}$ in the Theorem 3.2.4 for untwisted theory on X equal to those of the twisted theory by $N_{W/X}$ on W.*

Proof. Observe that $TW \oplus N_{W/X}$ and $N_{Z_j/W} \oplus N_{W/X}|_{Z_j}$ can be deformed to $TX|_W$ and $N_{Z_j/X}$, respectively. Hence they represent the same elements in the K-groups. All inputs involved in (i-iii) for $A_{i,k;j,j',\beta,\chi}$ are therefore identified. □

The proof of Theorem 3.2.4 will occupy the next subsection.

3.2.2 Proof of Theorem 3.2.4

The following proof is parallel to the proof of Givental's theorem in [5, Theorem 2].

For notational convenience, introduce

$$(ev_+)_*^{\mathrm{vir}}(\alpha) := \mathrm{PD}\left[(ev_+)_*(\alpha \cap [\overline{\mathcal{M}}_{j,j',\beta,\chi}]^{vir})\right], \tag{3.10}$$

where PD is the Poincaré duality map (The case for ev_- on $\overline{\mathcal{M}}_{j,j',\beta,\chi}$ or ev_i on $\overline{\mathcal{M}}_{0,n}(X,\beta)$ should be understood accordingly).

Recall $F(-z) \in \mathcal{L}_E$, and $F^j(-z) := \iota_j^* F(-z)$ is, by definition,

$$F^j(-z,t) = -1z + \iota_j^* t + \iota_j^* \sum_{n,d} \frac{Q^d}{n!} (\mathrm{ev}_1)_*^{\mathrm{vir}} \left[\frac{e_T^{-1}(E_{0,n+1,d})}{-z-\psi_1} \prod_{i=2}^{n+1} \mathrm{ev}_i^* t(\psi_i) \right]. \tag{3.11}$$

Later the variable t in "$F^j(-z,t)$" will be suppressed and change of variables (for example $F^j(w)$) will occur only in the "$-z$" variable. We are going to evaluate the above expression by virtual localization.

Let $\vec{\Gamma}$ be a decorated graph. and let $N_{\vec{\Gamma}}^{\mathrm{vir}}$ be the virtual normal bundle on $\overline{\mathcal{M}}_{\vec{\Gamma}}$. Then

$$F^j = -1z + \iota_j^* t + \sum_{\substack{\vec{\Gamma} \text{ with} \\ p_{s_1}=j}} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{Q^d}{n!} \iota_j^* (\mathrm{ev}_1)_*^{\mathrm{vir}} \left[\frac{e_T^{-1}(E_{0,n+1,d} \oplus N_{\vec{\Gamma}}^{vir})}{-z-\psi_1} \prod_{i=2}^{n+1} \mathrm{ev}_i^* t(\psi_i) \right], \tag{3.12}$$

where s_1 is the vertex where the first marking lies (see Definition 1.3.14). Given a decorated graph $\vec{\Gamma}$ with $\sum_{v\in V(\Gamma)} d_v = d$, write the contribution of $\vec{\Gamma}$ as

$$\mathrm{Cont}_{\vec{\Gamma}}(z) = \frac{1}{n!\mathrm{Aut}(\Gamma)} \iota_j^*(\mathrm{ev}_1)_*^{\mathrm{vir}} \left[\frac{e_T^{-1}(E_{0,n+1,d} \oplus N_{\vec{\Gamma}}^{vir})}{-z-\psi_1} \prod_{i=2}^{n+1} \mathrm{ev}_i^* t(\psi_i) \right]. \tag{3.13}$$

Since $\overline{\mathcal{M}}_{\vec{\Gamma}}$ is a fiber product of a collection of $\overline{\mathcal{M}}_{j,j',\beta,\chi}$ and $\overline{\mathcal{M}}_{0,E_v\cup S_v}(Z_{p_v}, d_v)$, this graph contribution can be computed by integrals on each of these spaces with suitable pull-backs and push-forwards. We will see in a moment that there is a recursive structure in the summation over graphs $\vec{\Gamma}$ such that $p_{s_1} = j$ (i.e., the first marked point lies on Z_j).

Consider first the contribution from graphs when the vertex s_1 is incident to a single edge with fractional character $\chi \in C(T)_{\mathbb{Q}}$. In this case, s_1 must be connected to another vertex v' such that $p_{v'} = j'$ for some $j' \neq j = p_{s_1}$. Let e be the edge connecting s_1 and v', $d_{s_1} := d_e \in \mathrm{NE}(X)$ be the degree associated to the edge e, and $\chi = \chi_{e,s_1}$ be the fractional character at s_1.

Introduce $t^{j,\chi}(z)$ as

$$\begin{aligned} t^{j,\chi}(z) &:= \sum_{\substack{\vec{\Gamma} \text{ with} \\ d_{s_1}=0,\, \chi_{e,s_1}=\chi \\ p_{s_1}=j,\, n_{s_1}=0,\, \mathrm{val}(s_1)=1}} Q^{d_\Gamma} \mathrm{Cont}_{\vec{\Gamma}}(z) \\ &= \sum_{j'\neq j, d} \iota_j^*(\mathrm{ev}_+)_*^{vir} \left[\frac{Q^d \mathrm{Cont}_{(e,s_1)}}{-z-\psi_+ + \chi} \sum_{\substack{\vec{\Gamma} \text{ with } d_{s_1}=0, \\ \chi_{e,s_1}\neq\chi, p_{s_1}=j'}} Q^{d_\Gamma} \left\{ (\mathrm{ev}_-)^* \mathrm{Cont}_{\vec{\Gamma}}(w) \right\}_{w=\psi_- - \chi} \right]. \end{aligned} \tag{3.14}$$

Expanding the sum over graphs, we get

$$t^{j,\chi}(z) = \sum_{j'\neq j,d} \iota_j^*(\mathrm{ev}_+)_*^{vir} \left[\frac{Q^d \mathrm{Cont}_{(e,s_1)}}{-z-\psi_+ + \chi} \left\{ (\mathrm{ev}_-)^* \left[F^{j'}(w) - t^{j',\chi}(w) \right] \right\}_{w=\psi_- - \chi} \right], \tag{3.15}$$

where the ev_+, ev_- are the maps from $\overline{\mathcal{M}}_{j,j',\beta,\chi}$, and the contribution

$$\mathrm{Cont}_{(e,s_1)} = \frac{e_T(\left[\pi_{j,j',\beta,\chi})_! T_{\pi_{j,j',\beta,\chi}}(-D_+)\right]^m)}{e_T(\left[(\pi_{j,j',\beta,\chi})_! f_{j,j',\beta,\chi}^*(TX \oplus E)\right]^m)}. \tag{3.16}$$

In the above, $T_{\pi_{j,j',\beta,\chi}}$ is the relative tangent bundle. D_+ is the divisor corresponding to the image of $s^+_{j,j',\beta,\chi}$. Superscript m means the moving part, i.e., the subsheaf generated by homogeneous elements of nontrivial T-character ([15]). In writing this, we also use the fact that $E_{0,n,d}$ does not have a fixed part due to the assumption that $E_{0,n,d}$ has invertible Euler class.

We interrupt the flow with a few remarks.

Remark 3.2.7. The reason to subtract $t^{j',\chi}(w)$ in (3.15) is to exclude the case when the other end at $Z_{j'}$ is directly connected to another unbroken map of the same fractional character χ (thus satisfying condition ($\star$) and violating the assignment of decorated graphs).

Remark 3.2.8. It might appear that there is a missing factor coming from the gluing at the node on the $s^-_{j,j',\beta,\chi}$ side. However, it is automatically taken care of because, in the expression $(\mathrm{ev}_-)^*\left[F^{j'}(w) - t^{j',\chi}(w)\right]$, the definition of $F^{j'}(w) - t^{j',\chi}(w)$ already involves the pull-back under ι_j and the pushforward under certain evaluation maps.

Remark 3.2.9. Since we pull back along $f_{j,j',\beta,\chi}$, the outcome only depends on the restriction $(TX \oplus E)|_{X_{j,j',\beta,\chi}}$. One can see the corresponding euler classes only depends on the class of $TX \oplus E$ in the $K^0(X_{j,j',\beta,\chi})$.

Now let us reorganize $t^{j,\chi}$. Plugging $w = \psi_- - \chi$ into $(ev_-)^*F^{j'}(w)$ in the expression (3.15), the term $(ev_-)^*F^{j'}(\psi_- - \chi)$ can be expanded using the Taylor expansion

$$
\begin{aligned}
(ev_-)^*F^{j'}(\psi_- - \chi) =&(ev_-)^*F^{j'}(-\chi) \\
&+ \psi_-(ev_-)^*\frac{\partial}{\partial t} F^{j'}(t)\Big|_{t=-\chi} + \frac{(\psi_-)^2}{2!}(ev_-)^*\frac{\partial^2}{\partial z^2} F^{j'}(t)\Big|_{t=-\chi} \\
&+ \cdots .
\end{aligned}
\tag{3.17}
$$

As ψ_- is nilpotent in $H^*((\overline{\mathcal{M}}_{j,j',\beta,\chi}))$, the above is a finite sum.

Applying the above Taylor expansion and expanding $\dfrac{1}{-z-\psi_+ + \chi}$ in terms of $(-z+\chi)^{-1}$, we have

$$
t^{j,\chi}(z) = \iota_j^*(ev_+)_*^{vir}\left[\sum_{i,k,j',d,\chi} \frac{Q^d A_{i,k;j,j',\beta,\chi}\, \frac{\partial^k}{\partial w^k}\left[(ev_-)^*(F^{j'}(w) - t^{j',\chi}(w))\right]\Big|_{w=-\chi}}{(-z+\chi)^i}\right], \tag{3.18}
$$

where $A_{i,j',k,d,\chi} \in H^*(\overline{\mathcal{M}}_{j,j',\beta,\chi})$ are some classes whose exact expressions do not concern us in this chapter. In a moment, we will show that all poles of F^j at $-\chi$ come from $t^{j,\chi}$, i.e., $\mathop{\mathbf{Prin}}\limits_{w=-\chi} F^j(w) = t^{j,\chi}(w)$.

Caution. There might be a confusion of signs. Notice we use $-z$ in $F^j(-z)$ but w in $F^j(w)$. However, $t^{j,\chi}(z)$ and $t^{j,\chi}(w)$ are used, where both z and w variables carry positive sign. As a result, the pole of $F^j(w)$ at $-\chi$ means the pole at $w=-\chi$, while $z=\chi$. This choice of signs is dictated by the localization expression.

Next we consider the case when the vertex s_1 is incident with more than one edge. Write

$$t^j(z)=\iota_j^* t(z)+\sum_{\chi} t^{j,\chi}(z). \tag{3.19}$$

Later t^j will fit into (3.11). Let us start by applying virtual localization to (3.11):

$$\begin{aligned} & F^j(-z) \\ = & -1z+\iota_j^* t+\sum_{\vec{\Gamma} \text{ with } p_{s_1}=j} Q^{d_\Gamma}\mathrm{Cont}_{\vec{\Gamma}}(z) \\ = & -1z+\iota_j^* t+\sum_{\substack{d,n,\\ j_1,\dots,j_{n+1}}} \iota_j^*(\mathrm{ev}_1)_*^{\mathrm{vir}}\cdot \\ & \cdot\left[\frac{Q^d}{n!}\frac{\mathrm{Cont}_{s_1}}{(-z-\psi_1)}\prod_{i=2}^{n+1}(\mathrm{ev})_i^*\left\{\iota_j^* t(\psi_+-\chi)+\sum_{\substack{\vec{\Gamma} \text{ with } d_{s_1}=0,\\ p_{s_1}=j_i,\, n_{s_1}=0,\, \mathrm{val}(s_1)=1}} Q^{d_\Gamma}\mathrm{Cont}_{\vec{\Gamma}}(\psi_+-\chi)\right\}\right], \end{aligned} \tag{3.20}$$

where ev_i are the evaluation maps in $\overline{\mathcal{M}}_{0,n+1}(Z_j,d)$ and

$$\mathrm{Cont}_{s_1}=e_T^{-1}(E_{0,n+1,d}\oplus(N_j)_{0,n+1,d}). \tag{3.21}$$

This expression follows from localization computation by observing the following points. Firstly, the vertex s_1 contributes $e_T^{-1}((N_j)_{0,n+1,d})$ to the Euler class of virtual normal sheaf. Secondly, the contribution coming from smoothing the nodes is included in $\mathrm{Cont}_{\vec{\Gamma}}(\psi_1-\chi)$.

Since

$$t^j=\iota_j^* t+\sum_{\substack{\vec{\Gamma} \text{ with } d_{s_1}=0,\\ p_{s_1}=j, n_{s_1}=0, \mathrm{val}(s_1)=1}} Q^{d_\Gamma}\mathrm{Cont}_{\vec{\Gamma}}(z), \tag{3.22}$$

we have

$$F^j(-z)=-1z+t^j(z)+\sum_{n,d}\frac{Q^d}{n!}\iota_j^*(ev_1)_*^{vir}\left[\frac{e_T^{-1}(E\oplus(N_j)_{0,n+1,d})}{-z-\psi_1}\prod_{i=2}^{n+1}\mathrm{ev}_i^* t^j(\psi_1)\right]. \tag{3.23}$$

Remark 3.2.10. Notice in the above expression, $e_T^{-1}(E\oplus(N_j)_{0,n+1,d})$ only depends on the class $[E\oplus N_j]\in K_T^0(Z_j)$. Furthermore, since ψ_1 is nilpotent for a fixed degree of Novikov variables Q^d, the last term is a polynomial in z^{-1} since there are finitely many graphs of curve class $\leq d$.

Observe now the only term contributing poles not at $z = 0$ or $z = \infty$ is $t^j(z)$. In particular, the pole at $z = -\chi$ is of the following form

$$t^{j,\chi}(z)$$
$$=\iota_j^*(ev_+)_*^{vir}\left[\sum_{i,j',d,k,\chi}\frac{Q^d A_{i,k;j,j',\beta,\chi}\,\frac{\partial^k}{\partial w^k}\left[(ev_-)^*\left(F^{j'}(w)-\operatorname*{\mathbf{Prin}}_{z=-\chi}F^{j'}(w)\right)\right]\Big|_{w=-\chi}}{(-z+\chi)^i}\right] \tag{3.24}$$

This is exactly $\operatorname*{\mathbf{Prin}}_{z=-\chi} F^j(z)$ in Theorem 3.2.4.

Let us summarize what we have obtained so far. If $F \in \mathcal{L}_E$, there exists $F^j \in \mathcal{L}^j_{N_j \oplus E}$, satisfying (1),(2) in Theorem 3.2.4, such that when expanded in $1/z$, $\iota_j^* F = F^j$ as Laurent series in $1/z$. The coefficients $A_{i,j,j',k,\chi}$ only depend on the classes of vector bundle $(TX \oplus E)|_{X_{j,j',\beta,\chi}}$, $N_j \oplus E|_{Z_j}$ in their corresponding K-groups, $\overline{\mathcal{M}}_{j,j',\beta,\chi}$, $[\overline{\mathcal{M}}_{j,j',\beta,\chi}]^{vir}$ and their universal families $\mathcal{C}_{j,j',\beta,\chi}$. This is the forward implication of the theorem.

We now proceed to prove the converse. Given $\{F^j\}$ satisfying (1), (2) in Theorem 3.2.4, we set

$$F := \sum_{j=1}^{l} (\iota_j)_! \frac{F^j}{e_T(N_j)}. \tag{3.25}$$

It follows from localization that $F^j = \iota_j^* F$. Our goal is to show that $F \in \mathcal{L}_E$. On the other hand, there is always a point $F' \in \mathcal{L}_E$ such that $[F']_+ = [F]_+$. By previous localization computation, we can find F'^j such that $F'^j = \iota_j^* F'$ as formal functions. It follows that $[F'^j]_+ = [F^j]_+$. The subsequent argument shows that $[F^j]_\bullet$ *and* $[F^j]_-$ *are uniquely determined by* $[F^j]_+$ *and the recursion relation* (condition (2) of Theorem 3.2.4), *assuming* $F^j \in \mathcal{L}^j_{N_j \oplus E}$ (condition (1) of Theorem 3.2.4). Since F^j, F'^j both satisfy conditions (1) and (2) in Theorem 3.2.4, it follows that $[F^j]_\bullet = [F'^j]_\bullet$, $[F^j]_- = [F'^j]_-$, and therefore, $F = F'$. To highlight the function space we are working on, the following convention is adapted.

We work on the formal neighborhood of the space $\mathcal{H}^j$ with polarization, defined similarly to that of $\mathcal{H}$ in Convention 4. The decomposition $\mathcal{H}^j = \mathcal{H}^j_+ \oplus \mathcal{H}^j_-$ induces a decomposition $(\mathcal{H}^j, -1z) = (\mathcal{H}^j, -1z)_+ \oplus (\mathcal{H}^j, -1z)_-$.

To simplify the expressions, we abuse the notation in the following way

$$\mathcal{H}^j := (\mathcal{H}^j, -1z), \quad \mathcal{H}^j_+ := (\mathcal{H}^j, -1z)_+, \quad \mathcal{H}^j_- := (\mathcal{H}^j, -1z)_-. \tag{3.26}$$

Since $\{F^j\}$ lie in the corresponding Lagrangian cones by assumption, there exists a unique $u^j \in H^j := H^*(Z_j)$ such that $G^j(z) := S_{u^j}(-z)F^j(z) \in z\mathcal{H}^j_+$ for a given j. (See [5, §2.3].)

Notice that F^j can be recovered according to $S^*_{u^j}(z)G^j(z) = F^j(z)$. We proceed to show that there is a recursive relation for G^j determined by $[F^j]_+$.

First of all, we have the S matrix $S_{u^j}(-z) = 1 + O(1/z) \in 1 + \mathcal{H}^j_-$, a formal series in $1/z$ but nevertheless a *polynomial* in $1/z$ for each Q^d term. It follows from formal manipulation of formal series in z^{-1} that the Taylor polynomial of order K at $z = -\chi$ for the S-matrix can be written as

$$\begin{aligned} S_{u^j}(-z) = &\left(S_{u^j}(\chi) + \frac{\partial}{\partial z} S_{u^j}(\chi)(-z-\chi) + \cdots + \frac{1}{K!}\frac{\partial^K}{\partial z^K} S_{u^j}(\chi)(-z-\chi)^K \right) \\ &+ (-z-\chi)^{K+1} R(z^{-1}), \end{aligned} \tag{3.27}$$

where $R(z^{-1}) \in \mathcal{H}^j_-$ is the remainder term.

Setting

$$\begin{aligned} \bar{P}^j_{i,\chi} =& \iota^*_j (ev_+)^{vir}_* \Bigg[\sum_{j',k,d} Q^d A_{i,k;j,j',\beta,\chi} \cdot \\ &\cdot \frac{\partial^k}{\partial z^k} \left[(ev_-)^* \left[S^*_{u^{j'}}(z) G^{j'}(z) - \underset{z=-\chi}{\mathbf{Prin}} \left(S^*_{u^{j'}}(z) G^{j'}(z) \right) \right] \right] \Bigg|_{z=-\chi} \Bigg], \end{aligned} \tag{3.28}$$

we have

$$\underset{z=-\chi}{\mathbf{Prin}}\, F^j(z) = \sum_{i=0}^{\infty} \frac{\bar{P}^j_{i,\chi}}{(z+\chi)^i}. \tag{3.29}$$

Notice that this sum has finite terms for any fixed power of Novikov variables. That is, the coefficient of Q^d has finite order pole . Therefore, it lives in the function space $\mathcal{H}^j$. The following is the recursion relation of G^j promised earlier.

Lemma 3.2.11. *We have the following recursion relation of $G^j(z)$*

$$G^j(z) = [G^j(z)]_+ + \sum_{\chi} \left(\sum_{i=0}^{\infty} \frac{\bar{P}^j_{i,\chi}}{(z+\chi)^i} \sum_{l=0}^{i-1} \frac{\partial^i}{\partial z^l} S_{u^j}(\chi)(-z-\chi)^l \right). \tag{3.30}$$

Notice that this becomes a finite sum for fixed Novikov variables.

Proof. The sum $\sum_{l=0}^{i-1} \frac{\partial^i}{\partial z^l} S_{u^j}(\chi)(-z-\chi)^l$ is the Taylor polynomial for $S(-z)$ of order $i-1$. Now use the above fact that $S_{u^j}(-z) - \sum_{l=0}^{i-1} \frac{\partial^i}{\partial z^l} S_{u^j}(\chi)(-z-\chi)^l = (-z-\chi)^i R(z^{-1})$. Simple calculation shows that the difference of the LHS and of the RHS of (3.30) is a z^{-1} series (and a polynomial in z^{-1} in each coefficient of Q^d with fixed d). However, since both sides lie in $\mathcal{H}^j_+$, so does the difference. Hence the difference must vanish. $\square$

This gives us a recursive relation for $G^j(z)$ with initial condition $[G^j(z)]_+ \in H^*(Z_j; S_T)[z][\![Q]\!]$ and $u^j \in H^j$. We note that the $[G^j(z)]_+$ term is denoted by $q^\alpha(z)$ in the proof of [5, Theorem 2].

The rest of the proof is identical to the corresponding part of *loc. cit.*. Setting $Q = 0$, $\iota_j^* t$ are related to $[G^j(z)]_+$ in the following way

$$-z + \iota_j^* t = \left[S_{u^j}^{-1}(-z)[G^j(z)]_+ \right]_+ . \tag{3.31}$$

We are left to show $\{u^j\}$ are determined by $\{\iota_j^* t\}$ in view of the fact that $G^j(z) \in z\mathcal{H}_+^j$, cf. [5, §2.3]. Again modulo Novikov variables, we have $G^j(z) = [G^j(z)]_+$, $S_{u^j}(-z) = e^{-u^j/z}$.

$$[G^j(z)]_+ = \left[e^{-u^j/z}(z + \iota_j^* t(-z)) \right]_+ \in z\mathcal{H}_+^j. \tag{3.32}$$

The constant term being zero provides us with an equation between u^j and $\iota_j^* t(z)$. Using formal implicit function theorem, one can see that u^j is uniquely determined by successive Q-adic approximations.

3.3 Gromov–Witten invariants of projective bundles

Let Y be a nonsingular variety equipped with an algebraic action by $T = (\mathbb{C}^*)^m$.

Definition 3.3.1. Y is said to be an *algebraic GKM manifold* if it has finitely many fixed points and finitely many one-dimensional orbits under the T action.

In this section, we assume Y to be a proper algebraic GKM manifold. In this case, the closure of any one-dimensional orbit in Y is $\mathbb{P}^1$. Examples include proper toric varieties, partial flag varieties including Grassmannians, etc.

Recall we have the following theorem stated in section 3.1.

Theorem 3.3.2. *The isomorphism $\mathfrak{F}$ between equivariant cohomology rings and numerical curve classes induces an isomorphism of the full genus zero equivariant Gromov–Witten invariants between $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$. More precisely,*

$$\langle \psi^{a_1}\sigma_1, \dots, \psi^{a_n}\sigma_n \rangle_{0,n,\beta} = \langle \psi^{a_1}\mathfrak{F}\sigma_1, \dots, \psi^{a_n}\mathfrak{F}\sigma_n \rangle_{0,n,\mathfrak{F}\beta}. \tag{3.33}$$

The rest of this section is devoted to the proof of this theorem. By definition of algebraic GKM manifolds, the fixed loci of $\mathbb{P}(V_i)$ are the fibers over the T-fixed points of Y. Let p be a fixed point in Y. For a given $1 \leq j \leq l$, Since $c_T(V_1) = c_T(V_2)$, we have

$$c_T(V_1|_p) = c_T(V_2|_p) \in R_T = H_T^*(\{\text{point}\}), \tag{3.34}$$

where $V_i|_p$ are the fibers at p. Interpreting R_T as the representation ring of T, the equality is also saying that $V_i|_p$ are isomorphic as T-representations. Thus $\mathbb{P}(V_1)|_p, \mathbb{P}(V_2)|_p$ are T-equivariantly isomorphic. Hence, the fixed loci of $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$ are naturally identified.

Given $Z \subset \mathbb{P}(V_1)|_p, Z' \subset \mathbb{P}(V_2)|_p$ two fixed loci that are identified under $\mathfrak{F}$. One sees that there is a T-equivariant short exact sequence

$$0 \to N_{Z/(\mathbb{P}(V_1)|_p)} \to N_{Z/\mathbb{P}(V_1)} \to \pi^* N_{\{p\}/Y} \to 0.$$

Thus, $N_{Z/\mathbb{P}(V_1)}$ can be T-equivariantly deformed to $N_{Z/(\mathbb{P}(V_1)|_p)} \oplus \pi^* N_{\{p\}/Y}$, where $\pi : \mathbb{P}(V_1) \to Y$ is the projection, by sending the T-equivariant extension class to zero. Similar observations conclude that $N_{Z/\mathbb{P}(V_1)}$ and $N_{Z'/\mathbb{P}(V_2)}$ represent the same element in $K_T^0(Z) \cong K_T^0(Z')$.

Now in order to identify Lagrangian cones $\mathcal{L}_{\mathbb{P}(V_1)}$ and $\mathcal{L}_{\mathbb{P}(V_2)}$, it suffices to show that the recursion relations (2) in Theorem 3.2.4 are identified under $\mathfrak{F}$. Let us analyze the one-dimensional orbit closures in Y first.

Lemma 3.3.3. *Let $p \in Y$ be a fixed point and l_1, l_2 be two distinct 1-dimensional orbit closures passing through p. The (integral) characters χ_1, χ_2 of l_1, l_2 at p do not lie on the same ray in $C(T)_{\mathbb{Q}}$.*

Proof. Suppose we have these two orbit closures l_1, l_2 such that χ_1, χ_2 lie on the same ray in $C(T)_{\mathbb{Q}}$. Write χ_0 to be the first $\mathbb{Z}$-point along the ray of χ_1 and χ_2 in $C(T)_{\mathbb{Q}}$. Decompose T_pX into irreducible T-representations $\bigoplus_{\chi \in C(T)} T_\chi$. The assumption implies that $\bigoplus_{a \in \mathbb{Z}_{\geq 0}} T_{a\chi_0}$ is at least of dimension 2.

Choosing ρ such that $(\rho, \chi_0) > 0$, there is a subvariety of Z^ρ containing p that is T-isomorphic to $\bigoplus_{(\rho,\chi)>0} T_\chi$ by [4, Theorem 2.5] (note that p is a discrete fixed point). Z^ρ contains the subrepresentation $\bigoplus_{a \in \mathbb{Z}_{\geq 0}} T_{a\chi_0}$. However, an elementary verification shows that there are infinitely many one-dimensional orbits in $\bigoplus_{a \in \mathbb{Z}_{\geq 0}} T_{a\chi_0}$ since its dimension is at least 2. This is a contradiction to the assumption that Y is algebraic GKM. $\square$

From now on, we will be mostly concerned with general properties of a projective bundle over an algebraic GKM manifold. To simplify the notation, set $\pi : X := \mathbb{P}_Y(V) \to$

Y. Recall the notations defined in section 1.3. $Z_1, \ldots, Z_n$ are the fixed loci of X under T-action, and $X_{j,j',\beta,\chi} = f_{j,j',\beta,\chi}(\mathcal{C}_{j,j',\beta,\chi})$ can be seen as the union of the images of all unbroken maps for fixed j, j', β, χ. (See Notations 1.)

Lemma 3.3.4. *Fix $\chi \in C(T)_{\mathbb{Q}}$ and $Z_j \subset X$ such that $\pi(Z_j) = \{p_j\} \subset Y$. For any $d \in \mathrm{NE}(X)$ and $Z_{j'}$, we have either $\overline{\mathcal{M}}_{j,j',\beta,\chi} = \emptyset$ or $\pi \circ f_{j,j',\beta,\chi}(\mathcal{C}_{j,j',\beta,\chi}) \subset l$, where $l \subset Y$ is the closure of a one-dimensional orbit.*

Proof. Fix χ and Z_j and pick any $d, Z_{j'}$. If $\overline{\mathcal{M}}_{j,j',\beta,\chi} \neq \emptyset$, $\pi(X_{j,j',\beta,\chi})$ must be either the fixed point $\{p_j\}$ or some one-dimensional orbit closure $l \subset Y$. In the latter case, by definition, χ must be proportional to the fractional character of l at p_j. By the previous lemma, there is at most one orbit closure l passing through p_j such that its fractional character is proportional to χ. Since χ is fixed at the beginning, such a universal l can be easily found for all d and $Z_{j'}$. □

Let l be a one-dimensional orbit closure in Y and $X|_l := \pi^{-1}(l) \subset X$. With the help of the previous lemma, when there is a fixed χ and Z_j, we just found an $l \subset Y$ such that in the recursion condition for $\mathcal{L}_X$, all possible unbroken maps that are used in computing $\mathop{\mathbf{Prin}}\limits_{z=-\chi} F^j(z)$ are contained in $X|_l$. We are left to match the recursion conditions on $\mathop{\mathbf{Prin}}\limits_{z=-\chi} F^j(z)$ when we change the bundle from V_1 to V_2. For that purpose, we need to make sense of $\mathcal{L}_{X|_l, N_{X|_l/X}}$. We follow Remark 1.2.4 by adding an auxiliary $\mathbb{C}^*$ action that acts on $X|_l$ trivially, but on $N_{X|_l/X}$ by scaling. Let $R_{T\times\mathbb{C}^*} = R_T[x]$ where x is the equivariant parameter for the extra $\mathbb{C}^*$ action. Let $R_T[x, x^{-1}]\!]$ be the ring of Laurent series in x^{-1}. Let $\pi_{0,n,d} : \mathcal{C}_{0,n,d} \to \overline{\mathcal{M}}_{0,n}(X|_l, d)$ be the universal curve and $f_{0,n,d} : \mathcal{C}_{0,n,d} \to X$ be the universal stable map. Let

$$0 \to N^0 \to N^1 \to 0$$

be a two-term complex of locally free sheaves, whose cohomology are $R^i(\pi_{0,n,d})_* f^*_{0,n,d} N_{X|_l/X}$. Recall we define

$$e_{T\times\mathbb{C}^*}((N_{X|_l/X})_{0,n,d}) = \frac{e_{T\times\mathbb{C}^*}(N^0)}{e_{T\times\mathbb{C}^*}(N^1)} \in (H^*_T(\overline{\mathcal{M}}_{0,n}(X|_l, d))[x])_{\mathrm{loc}}, \tag{3.35}$$

where $(H^*_T(\overline{\mathcal{M}}_{0,n}(X|_l, d))[x])_{\mathrm{loc}}$ is the localization of $H^*_T(\overline{\mathcal{M}}_{0,n}(X|_l, d))[x]$ by inverting all monic polynomials in x. Embedding it into $H^*_T(\overline{\mathcal{M}}_{0,n}(X|_l, d) \otimes_{R_T} R_T[x, x^{-1}]\!]$ as a subspace renders $\mathcal{L}_{X|_l, N_{X|_l/X}}$ well defined under the extra $\mathbb{C}^*$ action. We will prove, in Proposition 3.3.6, that the $x = 0$ limit exists in the sense of Remark 1.2.4.

Since $N_{X|_l/X} = \pi^* N_{l/Y}$, we first analyze the T action on $N_{l/Y}$.

Lemma 3.3.5. *There is a splitting $T = T' \times \mathbb{C}^*$ such that T' acts on l trivially. Furthermore, there is no nontrivial T'-fixed subsheaf in $N_{l/Y}$.*

Proof. As we have seen in Corollary 1.3.4, the one-dimensional orbit is isomorphic to $\mathbb{C}^*$. Picking any point on it, we have an induced group homomorphism $T \to \mathbb{C}^*$. Let K be its kernel. $Hom(-, \mathbb{C}^*)$ is an exact functor among abelian groups because $\mathbb{C}^*$ is injective. $Hom(K, \mathbb{C}^*)$ decomposes into the product of a free abelian group and a torsion one. Therefore K decomposes into $K' \times K''$ where $K' \cong (\mathbb{C}^*)^{m-1}$ and K'' is a finitely generated torsion abelian group. K' is naturally embedded in K. Using K' instead of K, we have a short exact sequence

$$0 \to K' \to T \to \mathbb{C}^* \to 0.$$

Since K' fixes points in l, there is an induced action of the $\mathbb{C}^*$ on l. Now with K' being a sub-torus, T splits into $K' \times \mathbb{C}^*$ due to injectivity of K'. We have shown the first statement with $T' = K'$.

We now proceed to the second statement. First, observe that $N_{l/Y}$ can be decomposed into T'-eigensheaves:

$$N_{l/Y} = \bigoplus_{\chi \in C(T')} N_\chi, \tag{3.36}$$

such that each component is locally free. It therefore suffices to think about $N_{l/Y}|_{\{p\}}$ where p is a fixed point on l. Similar to Lemma 3.3.3, decompose T_pY into irreducible T-representations $\bigoplus_{\chi \in C(T)} T_\chi$. The tangent direction along l corresponds to one of the factor T_{χ_0} for some $\chi_0 \in C(T)$. This χ_0 is proportional to the character corresponding to $T \to \mathbb{C}$ from the above short exact sequence. Notice that $N_{l/Y}|_{\{p\}}$ is T-isomorphic to T_pY/T_{χ_0}. Since Y has isolated fixed point and one-dimensional orbits, $T_0 = 0$, T_{χ_0} is one-dimensional. Hence there is no fixed subspace in T_pY/T_{χ_0}. □

We are left to analyze $e_{T\times\mathbb{C}^*}((N_{X|_l/X})_{0,n,d})^{-1}$. Let $R_{T\times\mathbb{C}^*} = R_T[x]$ and $R_T[x, x^{-1}]]$ be the ring of Laurent series in x^{-1}.

By definition, $e_{T\times\mathbb{C}^*}((N_{X|_l/X})_{0,n,d})^{-1} \in (H_T^*(\overline{\mathcal{M}}_{0,n}(X|_l,d))[x])_{\mathrm{loc}}$.

Proposition 3.3.6. *$e_{T\times\mathbb{C}^*}((N_{X|_l/X})_{0,n,d})^{-1}$ has $x=0$ limit in $H_T^*(\overline{\mathcal{M}}_{0,n}(X|_l,d))\otimes_{R_T} S_T$ in the sense of Remark 1.2.4.*

Since we are working on the invariant subvariety $X|_l$, in the rest of this proof, we replace the whole variety X by $X|_l$, and still use $\overline{\mathcal{M}}_{\vec{\Gamma}}$, $\overline{\mathcal{M}}_{j,j',\beta,\chi}$, etc. for the corresponding notions in $X|_l$.

Let $\pi_{\vec{\Gamma}} : \mathcal{C}_{\vec{\Gamma}} \to \overline{\mathcal{M}}_{\vec{\Gamma}}$ be the universal family over $\overline{\mathcal{M}}_{\vec{\Gamma}}$ and $f_{\vec{\Gamma}} : \mathcal{C}_{\vec{\Gamma}} \to X_l$ be the universal stable map.

Lemma 3.3.7. *$R^i(\pi_{\vec{\Gamma}})_* f_{\vec{\Gamma}}^*(N_{X|_l/X})$ are locally free for $i=0,1$.*

Proof. This amounts to saying that, given an invariant stable map $f : C \to X_l$ with n markings assigned with a decorated graph $\vec{\Gamma}$, the dimensions of $H^i(C, f^*N_{X|_l/X})$ for $i=0,1$ only depend on the graph $\vec{\Gamma}$ (i.e., when the stable map varies in $\overline{\mathcal{M}}_{\vec{\Gamma}}$, these dimensions are constant).

Let v be a vertex in $\vec{\Gamma}$. Suppose v corresponds to a sub-curve C_v. Under the projection $\pi : X|_l \to l$, C_v has to map to a point. Therefore, $N_{X|_l/X}|_{C_v}$ is a trivial sheaf and the dimension of $H^0(C, f^*N_{X|_l/X})$ is the rank of $N_{X|_l/X}$. There is no H^1 since we are working on genus-0 curves. If v corresponds to a point in C, the dimension of H^0 is also the rank of $N_{X|_l/X}$.

Let e be an edge in $\vec{\Gamma}$ and C_e be the sub-curve corresponding to e. There are two possibilities of the image of $f(C_e)$ under the projection $\pi : X|_l \to l$. Recall $d_e \in \mathrm{NE}(X|_l)$ is the curve class corresponding to $f(C_e)$.

1. If $\pi_* d_e = 0$, $\pi\circ f(C_e)$ is a point. And the same thing as the vertex case happens.

2. If $\pi_* d_e \neq 0$, $\pi\circ f(C_e)$ is l. Recall C_e is a chain of $\mathbb{P}^1$. Let $C_1,\dots,C_k$ be the irreducible components of C_e. In this case, there is only one component C_i such that $\pi\circ f(C_i) = l$ (otherwise the fractional characters on the components violate C_e being an unbroken map). Suppose $\pi_* d_e = k[l]$. Then $\pi\circ f : C \to l$ is a degree k cover between $\mathbb{P}^1$'s. Thus, $H^i(C_i, f^*N_{X|_l/X}) = H^i(C_i, f^*\pi^*N_{l/Y})$ depends only on k and the bundle $N_{l/Y}$ over l. Other components than C_i are mapped to points under $\pi\circ f$ and do not

contribute extra dimensions to $H^i(C, f^*N_{X|_l/X})$.

The dimensions of $H^i(C, f^*N_{X|_l/X})$ can then be calculated by passing to the normalization of C. To sum it up, $\dim(H^i(C, f^*N_{X|_l/X}))$ depends only on the decorated graph $\vec{\Gamma}$ assigned to the stable map $f : C \to X_l$ and the normal bundle $N_{l/Y}$ on l. This implies the lemma we are proving. $\square$

As a result, we have

$$e_{T\times\mathbb{C}^*}((N_{X|_l/X})_{0,n,d})^{-1}\Big|_{\overline{\mathcal{M}}_{\vec{\Gamma}}} = \frac{e_{T\times\mathbb{C}^*}(R^1(\pi_{\vec{\Gamma}})_* f^*_{\vec{\Gamma}}(N_{X|_l/X}))}{e_{T\times\mathbb{C}^*}(R^0(\pi_{\vec{\Gamma}})_* f^*_{\vec{\Gamma}}(N_{X|_l/X}))} \tag{3.37}$$

We are left to prove this quotient has $x = 0$ limit.

Lemma 3.3.8. *$e_T(R^i(\pi_{\vec{\Gamma}})_* f^*_{\vec{\Gamma}}(N_{X|_l/X}))$ are invertible elements in $H^*(\overline{\mathcal{M}}_{\vec{\Gamma}}) \otimes_{\mathbb{C}} S_T$ (notice that we are without the $\mathbb{C}^*$ action).*

Proof. We have

$$H^*_T(\overline{\mathcal{M}}_{\vec{\Gamma}}) = R_T \otimes_{\mathbb{C}} H^*(\overline{\mathcal{M}}_{\vec{\Gamma}}). \tag{3.38}$$

One can decompose the Euler class into

$$e_T(R^i(\pi_{\vec{\Gamma}})_* f^*_{\vec{\Gamma}}(N_{X|_l/X})) = \sigma_0 \otimes 1 + \sum_{j\geq 1} \sigma_j \otimes \tau_j \in R_T \otimes_{\mathbb{C}} H^*(\overline{\mathcal{M}}_{\vec{\Gamma}}), \tag{3.39}$$

where $\sigma_j \in R_T$ and $\tau_j \in H^{>0}(\overline{\mathcal{M}}_{\vec{\Gamma}})$. If we can prove $\sigma_0 \neq 0$, the expression above becomes invertible after tensoring S_T. In order to do this, we will show that $R^i(\pi_{\vec{\Gamma}})_* f^*_{\vec{\Gamma}}(N_{X|_l/X})$ do not have nontrivial fixed subsheaves. It suffices to prove it fiberwise, and can be reduced to the following.

Lemma 3.3.9. *For any proper irreducible invariant curve $C \subset X|_l$, $H^i(C, N_{X|_l/X})$ does not have any nontrivial fixed subspace for $i = 0, 1$.*

Proof. It is because of the splitting in (3.36) and the fact that $N_{X|_l/X} = \pi^* N_{l/Y}$. There are two cases.

1. If $\pi(C) = l$, since π is equivariant, T' acts on C trivially. $N_{X|_l/X}\big|_C$ splits into eigensheaves where T' acts by scaling. Since $N_0 = 0$, the lemma follows.

2. If $\pi(C)$ is a fixed point, although T' might act on C nontrivially, the underlying vector bundle $N_{X|_l/X}\big|_C$ is trivial since it is a pull-back from a point. Only $H^0(C, N_{X|_l/X})$ can be nonzero, and it consists of constant sections. T' acts on it without trivial sub-representation again due to $N_0 = 0$.

Therefore, the lemma follows. $\square$

As a result, $\sigma_0 \neq 0$. $e_T(R^i(\pi_{\vec{\Gamma}})_* f_{\vec{\Gamma}}^*(N_{X|_l/X}))$ are invertible because $H^*(\overline{\mathcal{M}}_{\vec{\Gamma}})$ is finite dimensional. This proves Lemma 3.3.8. $\square$

Proof of Proposition 3.3.6. Now that Lemma 3.3.8 is proven, the $x = 0$ limit for

$$e_{T\times\mathbb{C}^*}((N_{X|_l/X})_{0,n,d})$$

can be readily constructed via localization formula. The limit is invertible because each piece $e_T(R^i(\pi_{\vec{\Gamma}})_* f_{\vec{\Gamma}}^*(N_{X|_l/X}))$ on each connected component of $\overline{\mathcal{M}}_{\vec{\Gamma}}$ is invertible. $\square$

Finally, we use the following fact.

Lemma 3.3.10. *Any two equivariant vector bundles over $\mathbb{P}^1$ with the same equivariant Chern classes can be connected by an equivariant family of vector bundles over a connected base.*

Proof. This lemma amounts to saying that the moduli stack of equivariant vector bundles with fixed Chern class over $\mathbb{P}^1$ is connected. This can be proven by, for example, adding a framing condition to form the fine moduli of framed equivariant vector bundles ([29]). This forms a fine moduli space which is a covering of the moduli stack of equivariant vector bundles with connected fibers. By the presentation for moduli of framed toric bundles given in [29], in the case of $\mathbb{P}^1$, the moduli space is just a product of flag varieties which are obviously connected. $\square$

To sum it up, Lemma 3.3.10 shows that $\mathbb{P}(V_1)|_l$ can be equivariantly deformed to $\mathbb{P}(V_2)|_l$. Thus they have the same equivariant Gromov–Witten theory. Moreover, $N_{\mathbb{P}(V_i)|_l/\mathbb{P}(V_i)}$ are obviously the pull-back of the same vector bundle $N_{l/Y}$ over l. As a result, the Lagrangian cones $\mathcal{L}_{\mathbb{P}(V_i)|_l, N_{\mathbb{P}(V_i)|_l/\mathbb{P}(V_i)}}$ are identified for $i = 1, 2$. Using Corollary 3.2.6 with $X = \mathbb{P}(V_i), W = \mathbb{P}(V_i)|_l$, we are finally able to identify the recursion relations (2) for any fixed Z_j and χ in Theorem 3.2.4 between points in $\mathcal{L}_{\mathbb{P}(V_1)}$ and $\mathcal{L}_{\mathbb{P}(V_2)}$. Theorem 3.3.2 is now proven.

CHAPTER 4

THE RECONSTRUCTION OF ALL GENUS INVARIANTS VIA MASTER SPACE

In this chapter, we apply a different method and answer the Question 1 in full.

4.1 Statement of the main theorem

Let S be a smooth projective variety. V_1, V_2 are vector bundles of rank r over S with

$$c(V_1) = c(V_2). \tag{4.1}$$

The set-up in section 3.1 applies to the nonequivariant situation as well. Similar to the definition in section 3.1, let $\mathfrak{F}$ be the natural isomorphism between the cohomology rings of $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$. In this chapter, we prove the following.

Theorem 4.1.1. *Let S be a smooth projective variety, and V_1, V_2 be two vector bundles on S. Let $X_1, X_2, \mathfrak{F}$ also be the same as above. Suppose $c(V_1) = c(V_2)$. Then we have the following equality between Gromov–Witten invariants.*

$$\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n\rangle^{X_1}_{g,n,\beta} = \langle \psi^{k_1}\mathfrak{F}(\alpha_1), \ldots, \psi^{k_n}\mathfrak{F}(\alpha_n)\rangle^{X_2}_{g,n,\mathfrak{F}(\beta)} \tag{4.2}$$

for any cohomology classes $\alpha_1, \ldots, \alpha_n \in H^*(X_1)$, any set of natural numbers $k_1, \ldots, k_n$, any curve class $\beta \in N_1(X_1)$ and any genus $g \in \mathbb{Z}_{\geq 0}$.

Remark 4.1.2. Recall (in section 3.1) we abuse the notations by writing $\mathfrak{F} : N_1(\mathbb{P}(V_1)) \cong N_1(\mathbb{P}(V_2))$ the natural isomorphism on curve classes. Later in section 4.4, we will involve the natural isomorphism on different projective bundles. To signify what projective bundles we are on, we will also write this isomorphism as $\mathfrak{F}_{V_1,V_2} : N_1(\mathbb{P}(V_1)) \cong N_1(\mathbb{P}(V_2))$.

Let us make a preliminary observation towards the theorem.

Lemma 4.1.3. *We have the equality $(K_{X_1}, \beta) = (K_{X_2}, \mathfrak{F}(\beta))$.*

As a result, the virtual dimensions of the involved moduli spaces of stable maps are the same, which is one of the first things one can check directly for the theorem.

Proof. Recall the Euler sequence

$$0 \to \mathcal{O}_{X_i} \to \pi_i^* V_i \otimes \mathcal{O}_{X_i}(1) \to T_{\pi_i} \to 0,$$

where the T_{π_i} is the relative tangent sheaf of the morphism π_i. We also know that for $i = 1, 2$

$$K_{X_i} = K_{\pi_i} \otimes \pi_i^* K_S = det(T_{\pi_i}^\vee) \otimes \pi_i^* K_S. \tag{4.3}$$

Thus, it suffices to prove that

$$\mathfrak{F}(c_1(det(T_{\pi_1}^\vee))) = c_1(det(T_{\pi_2}^\vee)). \tag{4.4}$$

We see from the Euler sequence that

$$c(T_{\pi_i}) = c(\pi_i^* V_i \otimes \mathcal{O}_{X_i}(1)) \tag{4.5}$$

for $i = 1, 2$. The proof can be easily finished with a splitting principle calculation using the fact that $\mathfrak{F} \circ \pi_1^* = \pi_2^*$, $\mathfrak{F}(c_1(\mathcal{O}_{X_1}(1))) = c_1(\mathcal{O}_{X_2}(1))$ and $c(V_1) = c(V_2)$. □

Note that in proving the theorem, we are free to twist both V_1 and V_2 by a fixed line bundle *simultaneously*. To be more precise, for any line bundle L on S, we still have

$$c(V_1 \otimes L^{-1}) = c(V_2 \otimes L^{-1}) \tag{4.6}$$

and

$$\mathbb{P}(V_i \otimes L^{-1}) \cong \mathbb{P}(V_i). \tag{4.7}$$

Throughout this paper, we choose L to be sufficiently ample so that $\mathcal{O}_{\mathbb{P}(V_i \otimes L^{-1})}(1) = \mathcal{O}_{\mathbb{P}(V_i)}(1) + \pi_i^* L$ is ample on X_i. Therefore, without loss of generality,

Assumption 1. *We assume $\mathcal{O}_{\mathbb{P}(V_i)}(1)$ to be ample on $\mathbb{P}(V_i)$ for $i = 1, 2$.*

Later, more specific requirement about the ampleness of L will be made.

4.2 Virtual localization

In order to carry out the recursive algorithm in section 4.3, we need to use virtual localization in section 1.3 again. The virtual localization is applied in a very specific set-up as follows.

Let S be a smooth projective variety and V a vector bundle on S. Consider the space $\mathbb{P}(V \oplus \mathcal{O})$. For simplicity, we denote

$$X = \mathbb{P}(V \oplus \mathcal{O}). \tag{4.8}$$

There is a natural inclusion

$$\mathbb{P}(V) \hookrightarrow X$$

whose complement can be naturally identified as the total space of the vector bundle V, i.e.,

$$V \cong X - \mathbb{P}(V). \tag{4.9}$$

The $\mathbb{C}^*$ action on V by scaling extends to X in the obvious way. Under this $\mathbb{C}^*$ action, the fixed loci of X are

1. A copy of S (denoted by X_0) which can be identified as the zero section of V under the isomorphism in (4.9).

2. A copy of $\mathbb{P}(V)$ (denoted by X_∞) which can be identified as $X \backslash V$.

We denote $X^T = X_0 \bigcup X_\infty$. Their normal bundles are

1. $N_{X_0/X} = V$. There is an induced fiberwise $\mathbb{C}^*$ action of character 1 for any sub-representation.

2. $N_{X_\infty/X} = \mathcal{O}_{\mathbb{P}(V)}(1)$. There is an induced fiberwise $\mathbb{C}^*$ action of character -1 for any sub-representation.

Proposition 4.2.1. *The equivariant cohomology*

$$H^*_{\mathbb{C}^*}(X) = H^*(S)[\lambda, h]/(h^r + c_1(V)h^{r-1} + \cdots + c_r(V))(h - \lambda), \tag{4.10}$$

where $r = rank(V)$, λ is the equivariant parameter, and h is the equivariant cohomology class that restricts to λ on X_0 and to $c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ on X_∞.

We omit the standard computation.

4.2.1 Decorated graph

We associate decorated graphs to components of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ in the exact same way as described in section 1.3.3. Since our situation is more specific, we modify and add some notations of decorated graphs as follows.

1. The label map $\vec{p} : V(\Gamma) \to \{0, \infty\}$ whose choice depends on whether the image of $\mathfrak{c}_v$ lies in X_0 or X_∞.

2. Define n_v to be the number of markings on the component $\mathfrak{c}_v$

In addition, we introduce the following sets.

Definition 4.2.2. A vertex $v \in V(\Gamma)$ is called stable if $2g_v - 2 + val(v) + n_v > 0$ or $\beta_v \neq 0$. Let $V^S(\Gamma)$ be the set of stable vertices in $V(\Gamma)$. Let

$$V^1(\Gamma) = \{v \in V(\Gamma) \quad | \quad g_v = 0, val(v) = 1, n_v = 0, \beta_v = 0\}, \tag{4.11}$$

$$V^{1,1}(\Gamma) = \{v \in V(\Gamma) \quad | \quad g_v = 0, val(v) = n_v = 1, \beta_v = 0\}, \tag{4.12}$$

$$V^2(\Gamma) = \{v \in V(\Gamma) \quad | \quad g_v = 0, val(v) = 2, n_v = 0, \beta_v = 0\}. \tag{4.13}$$

The union of $V^1(\Gamma), V^{1,1}(\Gamma), V^2(\Gamma)$ is the set of unstable vertices.

Define an equivalence relation $\sim$ on the set $E(\Gamma)$ by setting $e_1 \sim e_2$ if there is a $v \in V^2(\Gamma)$ such that $e_1, e_2 \in E_v$.

Definition 4.2.3. Define $\overline{E}(\Gamma) = E/\sim$.

One easily sees that a class $[e] \in \overline{E}(\Gamma)$ consists of a chain of edges, say $e_1, e_2, \ldots, e_m$ such that e_i and e_{i+1} intersect at a $v_i \in V^2(\Gamma)$. There are also two vertices $v_0 \in e_1$ and $v_m \in e_m$ such that $v_0, v_m \notin V^2(\Gamma)$.

Definition 4.2.4. Define $V^2_{[e]} = \{v_1, \ldots, v_{m-1}\}$ and $V^{end}_{[e]} = \{v_0, v_m\}$.

Definition 4.2.5. Define $\overline{E}^{tail}(\Gamma)$ be the set of edge classes $[e] \in \overline{E}(\Gamma)$ such that $V^{end}_{[e]} \bigcap V^1(\Gamma) \neq \emptyset$ or $V^{end}_{[e]} \bigcap V^{1,1}(\Gamma) \neq \emptyset$.

Definition 4.2.6. Define $V^\infty(\Gamma) = \{v \in V(\Gamma) \quad | \quad p_v = \infty\}$ and $V^0(\Gamma) = \{v \in V(\Gamma) \quad | \quad p_v = 0\}$.

Definitions 4.2.2-4.2.6 are used to describe certain summation indices in the virtual localization formula.

4.3 Computing invariants on a projective bundle

Let S be the smooth projective variety. In this section, we focus on a single vector bundle V. Write $\pi : \mathbb{P}(V) \to S$ to be the projection. Later in this section, we recursively establish an algorithm genus by genus. For each genus g, there are some additional data we need to choose before the recursion begins.

Lemma 4.3.1. *For any $g \in \mathbb{N}$, there is a sufficiently ample line bundle $L_g \in Pic(S)$ such that for any $\beta \in \mathrm{NE}(\mathbb{P}(V))$ with $\pi_*\beta \neq 0$, we have the intersection pairing*

$$(\beta, \mathcal{O}_{\mathbb{P}(V)}(1) + \pi^* L_g) > max\{g-1, 0\}. \tag{4.14}$$

Proof. For an ample L, we always have $(\overline{\beta}, L) \geq 1$ for any nonzero $\overline{\beta} \in \mathrm{NE}(S)$. By replacing L by its multiple mL, we can assume that $(\overline{\beta}, L) \geq m$ holds for any integer m and any nonzero $\overline{\beta} \in \mathrm{NE}(S)$. Back to the lemma, for any $\beta \in \mathrm{NE}(\mathbb{P}(V))$, notice that

$$(\beta, \mathcal{O}_{\mathbb{P}(V)}(1) + \pi^* L_g) = (\beta, \mathcal{O}_{\mathbb{P}(V)}(1)) + (\beta, \pi^* L_g) = (\beta, \mathcal{O}_{\mathbb{P}(V)}(1)) + (\pi_*\beta, L_g), \tag{4.15}$$

where the last equality follows from the projection formula and the last intersection pairing $(\pi_*\beta, L_g)$ is evaluated in S. Since $g-1$ is a fixed integer, the inequality can be achieved by choosing L to be a sufficiently high multiple of an ample line bundle. $\square$

Now that the collection of ample line bundles $\{L_g\}_{g\in\mathbb{N}}$ are chosen, we are ready to establish a recursive algorithm.

Theorem 4.3.2. *Let $f \in N_1(\mathbb{P}(V))$ be the class of a line in a fiber. Suppose the fiber integrals $\langle \cdots \rangle^{\mathbb{P}(V)}_{g,n,kf}$ are known. There is an algorithm determining genus g untwisted invariants of $\mathbb{P}(V)$ from the genus g' twisted invariants of S (twisted by $V \otimes L_{g''}$ where $g'' \leq g$) such that $g' \leq g$. Furthermore, besides the twisted invariants of S, this algorithm only depends on the cohomology rings $H^*(S)$ and $H^*(\mathbb{P}(V))$, the cohomology class $c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$, the group of numerical curve classes $N_1(\mathbb{P}(V))$ and its intersection pairing with $H^2(\mathbb{P}(V))$, the pull-back morphism $\pi^* : H^*(S) \to H^*(\mathbb{P}(V))$, and the Mori cone $\mathrm{NE}(\mathbb{P}(V))$.*

We rule out the fiber integrals $\langle \cdots \rangle^{\mathbb{P}(V)}_{g,n,kf}$ because they serve as the initial case for an induction, obtained by a method different from localization. To sum it up, the structure of this section is as follows. We determine fiber integrals in section 4.3.1 in a way that's good enough for the purpose of Theorem 4.1.1. And we prove Theorem 4.3.2 in sections 4.3.2, 4.3.3, and 4.3.4 by using fiber integral as the initial case of the induction.

4.3.1 Fiber classes

In this section, we determine integrals of the form $\langle \cdots \rangle^{\mathbb{P}(V)}_{g,n,kf}$ where $f \in N_1(\mathbb{P}(V))$ is the line class in a fiber. Similar to [27, section 1.2], $\overline{\mathcal{M}}_{g,n}(\mathbb{P}(V), kf)$ is fibered over S, i.e., there is a morphism

$$p : \overline{\mathcal{M}}_{g,n}(\mathbb{P}(V), kf) \to S.$$

In fact, $\overline{\mathcal{M}}_{g,n}(\mathbb{P}(V), kf) = \mathcal{P} \times \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, k)/GL(r+1)$ where $\mathcal{P}$ is the principal $GL(r+1)$ bundle corresponding to V, and $GL(r+1)$ acts on the product diagonally. We can decompose the virtual class as

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}(V), kf)]^{vir} = e(\mathbb{E} \boxtimes T_S) \cap [\overline{\mathcal{M}}_{g,n}(\mathbb{P}(V), kf)]^{vir_p}, \tag{4.16}$$

where $\mathbb{E}$ is the Hodge bundle and $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}(V), kf)]^{vir_p}$ is the relative virtual class. This reduces our problem to the determination of integrals against $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}(V), kf)]^{vir_p}$. We can describe this relative virtual class in terms of something that is well-understood.

Without breaking Assumption 1, we can replace V by $V \otimes L^{-1}$ where L is sufficiently ample. As a result, we can assume $V^\vee$ is globally generated. Thus, there is a surjection map $\mathcal{O}^N \to V^\vee$ for some large integer N. By taking the dual, we embed V into a trivial bundle

$$V \hookrightarrow \mathcal{O}^N.$$

By universal property of Grassmannian, it induces a morphism

$$f : S \to Gr(r+1, N)$$

such that $f^*U = V$ where U is the tautological bundle of rank $r+1$. Write $G = Gr(r+1, N)$ in short. In the meantime, we have the $\overline{\mathcal{M}}_{g,n}(\mathbb{P}_G(U), kf)$ whose base change via f is isomorphic to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}(V), kf)$. Now we have the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n}(\mathbb{P}_S(V),kf) \arrow[r, "\bar{f}"] \arrow[d, "p"] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}_G(U),kf) \arrow[d, "\bar{p}"] \\
S \arrow[r, "f"] & G
\end{tikzcd}$$

It is easy to see in fact $\bar{f}^![\overline{\mathcal{M}}_{g,n}(\mathbb{P}_G(U),kf)]^{vir_{\bar{p}}} = [\overline{\mathcal{M}}_{g,n}(\mathbb{P}_S(V),kf)]^{vir_p}$. As a result,

$$\bar{p}^* f_!\sigma \cap [\overline{\mathcal{M}}_{g,n}(\mathbb{P}_G(U),kf)]^{vir_{\bar{p}}} = \bar{p}^*\sigma \cap \bar{f}_![\overline{\mathcal{M}}_{g,n}(\mathbb{P}_S(V),kf)]^{vir_p}, \tag{4.17}$$

where $\sigma \in H^*(S)$.

An insertion is in the form $h^i\pi^*\sigma \in H^*(\mathbb{P}_S(V))$ where $h = c_1(\mathcal{O}_{\mathbb{P}_S(V)}(1))$, $\sigma \in H^*(S)$. It corresponds to a factor $ev_j^*(h^i\pi^*\sigma) \in H^*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}_S(V),kf))$ in the integrand. Notice the fact that $ev_j^*\pi^*\sigma = p^*\sigma$, and $ev_j^*h = \bar{f}^*(ev_j^*h')$ where $h' = c_1(\mathcal{O}_{\mathbb{P}_G(U)}(1))$. In view of all of these, we can push the integrands against $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}_S(V),kf)]^{vir_p}$ forward and reduce it to an integral against $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}_G(U),kf)]^{vir_{\bar{p}}}$. On the other hand, $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}_G(U),kf)]^{vir}$ and $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}_G(U),kf)]^{vir_{\bar{p}}}$ are well-understood since we can use the localization on the Grassmannian.

This method is enough for us to establish our main theorem in the special cases of fiber classes.

Lemma 4.3.3. *Theorem 4.1.1 holds if $\beta = kf$ for some $k \in \mathbb{Z}_{>0}$.*

Proof. Given V, V' such that $c(V) = c(V')$, we can twist them by the inverse of a sufficiently ample line bundle to carry out all the above constructions at the same time. In particular, we have morphisms

$$f, f' : S \to G$$

for some N such that $f^*U = V$ and $f'^*U = V'$. Notice that the Chern classes $c_i(U)$ generates the cohomology ring $H^*(G)$. Because $f^*c_i(U) = f'^*c_i(U)$, we conclude that f, f' induce the same pull-back morphisms between cohomology rings, i.e., $f^* = f'^* \in Hom(H^*(G), H^*(S))$. As a result, $f_!(\sigma) = f'_!(\sigma)$ for any $\sigma \in H^*(S)$. Therefore, if we carry out all the above reductions for V and V' in parallel, and push the integrands forward into $\overline{\mathcal{M}}_{g,n}(\mathbb{P}_G(U),kf)$, we get the same integrals. $\square$

4.3.2 Set-ups for the main algorithm

Starting from this subsection, we aim at proving Theorem 4.3.2. We say

$$\beta < \beta' \in \mathrm{NE}(X)$$

if either $(\pi_X)_*\beta' - (\pi_X)_*\beta \in \mathrm{NE}(S)$ or $\beta' - \beta \in \mathrm{NE}(X))$. We are ready to state the induction hypothesis.

Induction hypothesis. *We induct on the genus and the numerical curve classes. Fix a genus g_0 and an effective curve class $\beta_0 \in \mathrm{NE}(\mathbb{P}(V))$. For any g, β such that $\beta < \beta_0$ and $g \le g_0$, assume that the formulas of invariants of the form $\langle \dots \rangle^{\mathbb{P}(V)}_{g,n,\beta}$ in terms of the ones of the form $\langle \dots \rangle^{S,tw,V\otimes L^{-1}_{g'}}_{g',n',\overline{\beta}'}$ (where $g' \le g$) are found.*

Since fiber classes are already determined in the previous section, we may assume the following.

Assumption 2. *For the rest of the section, we assume $\pi_*\beta_0 \neq 0$.*

Goal. *Express $\langle \psi^{k_1}\alpha_i, \dots, \psi^{k_n}\alpha_n \rangle^{\mathbb{P}(V)}_{g_0,n,\beta_0}$ in terms of Gromov–Witten invariants of the form $\langle \cdots \rangle^{\mathbb{P}(V)}_{g,n',\beta}$ with $\beta < \beta_0$ and $g \le g_0$, plus the twisted invariants of S.*

From now on, the genus g_0 is fixed and the line bundle L_{g_0} with index g_0 is used throughout the rest of the section. We consider the Gromov–Witten invariants with target $\mathbb{P}_S(V \otimes L^{-1}_{g_0} \oplus \mathcal{O})$. To simplify notations, we can replace V by $V \otimes L^{-1}_{g_0}$ just as in Assumption 1, which leads to the following without loss of generality.

Assumption 3. *For the rest of the section, in addition to the induction hypothesis and the ampleness of $\mathcal{O}_{\mathbb{P}_S(V)}(1)$, we also assume $(\beta, \mathcal{O}_{\mathbb{P}_S(V)}(1)) > g_0 - 1$ for any $\beta \in \mathrm{NE}(\mathbb{P}_S(V))$ such that $\pi_*\beta \neq 0$.*

Recall that in the previous section, we wrote $X = \mathbb{P}(V \oplus \mathcal{O})$ and introduced X_∞, X_0 along with other notations. We continue to use those notations. Recall there are the isomorphisms

$$\mathbb{P}(V) \cong X_\infty \subset X, \quad S \cong X_0 \subset X. \tag{4.18}$$

We can apply virtual localization to certain invariants on X to obtain relations of invariants on $\mathbb{P}(V)$ and the ones on S. In order to write down the localization formula, we need to fix some notations.

Notation. 1. *To simplify notations, we write $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}(V)}(1)$ and $H = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$.*

2. *Given a decorated graph Γ, for any $v \in \Gamma$, we label the markings on the component $\mathfrak{c}_v$ by $m_1^v, \ldots, m_{n_v}^v$.*

3. *Choose a $\mathbb{C}$-basis $\{T_i\}$ for the cohomology ring $H^*(S)$ with dual basis written as $\{T^i\}$ (under Poincaré pairing).*

Besides, the following convention is used throughout the rest of the section.

Convention 5. *Let $\iota : X_\infty \hookrightarrow X$ be the inclusion. Since we have the natural identification of the numerical curve classes $\iota_* : N_1(X_\infty) \cong N_1(X)$, we won't distinguish curve classes in X and X_∞ notation-wise. Furthermore, given $\beta \in N_1(X_\infty)$, since $(\beta, \mathcal{O}_{X_\infty}(1)) = (\iota_*\beta, \mathcal{O}_X(1))$, we write $(\beta, \mathcal{O}(1))$ for this intersection pairing without distinguishing whether it is evaluated on X_∞ or X.*

4.3.3 Lifting of insertions

Given a cohomology class $\alpha \in H^*(X_\infty)$, let $\tilde{\alpha}$ be a lifting of α to $H^*_{\mathbb{C}^*}(X)$ defined in the following way:

α can always be written as

$$\alpha = c_1(\mathcal{O}(1))^e \cup \pi^*\overline{\alpha} \in H^*(X_\infty), \tag{4.19}$$

where $\overline{\alpha} \in H^*(S)$. Under the presentation in Proposition 4.2.1, $c_1(\mathcal{O}(1))$ has a lifting h. Define

$$\tilde{\alpha} = h^e \cup \pi_X^*\overline{\alpha} \in H^*_{\mathbb{C}^*}(X), \tag{4.20}$$

where $\pi_X : X \cong \mathbb{P}(V \oplus \mathcal{O}) \to S$ is the projection. Obviously $\tilde{\alpha}$ restricts to α on X_∞ and to $\lambda^e \pi^*\overline{\alpha}$ on X_0.

4.3.4 Localization formula

Let us introduce some notations.

Notation. 1. *Define*

$$\mathbf{c}_V(x) = x^r + c_1(V)x^{r-1} + \cdots + c_r(V), \tag{4.21}$$

where $r = rank(V)$.

2. *When we fix a class $[e] \in \bar{E}(\Gamma)$, we denote v_+, v_- to be the two vertices in $V^{end}_{[e]}$ (whichever is arbitrary).*

3. *Define $\iota_\pm : \mathbb{P}(V) \to X_0$ to be the projection if $p_{v_\pm} = 0$, or $\iota_\pm : \mathbb{P}(V) \to X_\infty$ the identity map if $p_{v_\pm} = \infty$, respectively.*

4. *Let $e_\pm$ be the edge in the class $[e]$ that contains $v_\pm$, respectively (they can be the same).*

5. *For a vertex v, define $Vert(v) = H - \lambda, \delta_v = 1$ if $p_v = 0$, or $Vert(v) = \mathbf{c}_V(\lambda), \delta_v = -1$ if $p_v = \infty$, respectively.*

Consider the equivariant Gromov–Witten invariant

$$\langle \psi^{k_1}\tilde{\alpha}_1, \ldots, \psi^{k_n}\tilde{\alpha}_n \rangle^X_{g_0,n,\beta_0} \in \mathbb{C}[\lambda]. \tag{4.22}$$

Now let us write down the virtual localization formula:

$$\begin{aligned}
&\langle \psi^{k_1}\tilde{\alpha}_1, \ldots, \psi^{k_n}\tilde{\alpha}_n \rangle^X_{g_0,n,\beta_0} \\
=&\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n \rangle^{X_\infty,tw,\mathcal{O}(1)}_{g_0,n,\beta_0} + \\
&\sum_{\Gamma \neq \Gamma_0} \frac{1}{Aut(\Gamma)} \prod_{v \in V^S(\Gamma)} \sum_{\{i_{[e]}\}_{[e] \in \overline{E}(\Gamma) - \overline{E}^{tail}(\Gamma)}} \langle \psi^{k_{m_1^v}}\tilde{\alpha}_{m_1^v}|_{X_{p_v}}, \ldots, \psi^{k_{m_{n_v}^v}}\tilde{\alpha}_{m_{n_v}^v}|_{X_{p_v}}, \ldots \rangle^{X_{p_v},tw,\star}_{g_v,n,\beta_v}
\end{aligned} \tag{4.23}$$

A few explanations are in order:

- We sum over all decorated graphs with $g_\Gamma = g_0$ and $\beta_\Gamma = \beta_0$. Γ_0 is the decorated graph such that $V(\Gamma)$ consists of a single element v_0, $E(\Gamma) = \emptyset$, and $p_{v_0} = \infty$.

- We adopt the obvious convention that when $X_{p_v} = X_0$, the invariant is twisted by V (with fiberwise $\mathbb{C}^*$ action of character 1 for any sub-representation), and when $X_{p_v} = X_\infty$, the invariant is twisted by $\mathcal{O}(1)$ (with fiberwise $\mathbb{C}^*$ action of character -1). In other words, the $\star$ symbol in the superscript at the end of the last line needs to be replaced by either V or $\mathcal{O}(1)$ depending on the situation.

- Each $i_{[e]}$ in $\{i_{[e]}\}_{[e] \in \overline{E}(\Gamma) - \overline{E}^{tail}(\Gamma)}$ determines an element $T_{i_{[e]}}$ in the basis $\{T_i\}$. As suggested by the notation, they are indexed by $\overline{E}(\Gamma) - \overline{E}^{tail}(\Gamma)$.

- The last $\ldots$ sign in the twisted invariant should be inserted as follows. For any $[e] \in \overline{E}(\Gamma)$, we have $v_+, v_- \in V^{end}_{[e]}$ as before. Some insertions are inserted into $\langle \cdots \rangle^{X_{p_{v_\pm}},tw,*}_{g_{v_\pm},val(v_\pm),\beta_{v_\pm}}$ which are specified in the following two cases: (recall fiber integrals are treated differently. Therefore, at least one of $v_\pm$ has nontrivial degree.)

1. Suppose one of v_+ and v_- is in $V^1(\Gamma)$. Say $v_- \in V^1(\Gamma)$. Then an insertion
$$(\iota_+)_* \left(\frac{1}{k_{e_-}} \frac{Edge(\Gamma, [e])}{\delta_{v_+}(\lambda - H)/k_{e_+} - \psi} \right)$$
is inserted into the summand $\langle \cdots \rangle^{p_{v_+}, tw, *}_{g_{v_+}, val(v_+), \beta_{v_+}}$. As a convention, we take ψ to be a formal variable under the Gysin push-forward $(\iota_+)_*$, and then it is evaluated as the ψ-class in the corresponding Gromov–Witten invariant. The same convention is used in the rest of the section.

2. Otherwise, suppose one of v_+ and v_- is in $V^{1,1}(\Gamma)$. Say $v_- \in V^{1,1}(\Gamma)$. Then an insertion
$$(\iota_+)_* \left(\frac{1}{Vert(v_-)} \cdot \frac{Edge(\Gamma, [e])}{\delta_{v_+}(\lambda - H)/k_{e_+} - \psi} \cdot \psi^{k_{m_1^{v_-}}} (\iota_-)^* \left(\tilde{\alpha}_{m_1^{v_-}} | X_{p_{v_-}} \right) \right)$$
is inserted into the summand $\langle \cdots \rangle^{p_{v_+}, tw, *}_{g_{v_+}, val(v_+), \beta_{v_+}}$.

3. If none of the above applies, v_+, v_- must all be stable vertices. In this case, an insertion
$$(\iota_+)_* \left(\frac{Edge(\Gamma, [e]) T_{i_{[e]}}}{\delta_{v_+}(\lambda - H)/k_{e_+} - \psi} \right)$$
should be placed in the summand $\langle \ldots \rangle^{p_{v_+}, tw, *}_{g_{v_+}, val(v_+), \beta_{v_+}}$. In the meantime, an insertion
$$(\iota_-)_* \left(\frac{T^{i_{[e]}}}{\delta_{v_-}(\lambda - H)/k_{e_-} - \psi} \right)$$
should be placed in the $\langle \ldots \rangle^{p_{v_-}, tw, *}_{g_{v_-}, val(v_-), \beta_{v_-}}$ summand.

In all the above expressions, the edge contribution $Edge(\Gamma, [e])$ is equal to
$$\frac{\prod_{v \in V^2_{[e]} \cup V^{end}_{[e]}} Vert(v)}{\prod_{v \in V^2_{[e]}} \left(\left(\sum_{e' \in E_v} \frac{1}{k_{e'}} \right) \delta_v(\lambda - H) \right) \prod_{e \in [e]} \prod_{m=1}^{k_e} \left(\frac{m}{k_e}(H - \lambda) \mathbf{c}_V \left(H + \frac{m}{k_e}(\lambda - H) \right) \right)}.$$

Lemma 4.3.4. *In* (4.23)*, take α_i, k_i such that*
$$\sum_{i=1}^n k_i + \sum_{i=1}^n deg(\alpha_i) = dim[\overline{\mathcal{M}}_{g_0,n}(\mathbb{P}(V), \beta_0)]^{vir}. \tag{4.24}$$
If the curve class $\beta_0 \in N_1(X)$ satisfies $\beta_0 \in \mathrm{NE}(X)$ and $(\beta_0, \mathcal{O}(1)) > g_0 - 1$, the left-hand side of (4.23) *is zero.*

Proof. It can be proven by dimension counting. First of all, recall

$$\langle \psi^{k_1}\tilde{\alpha}_1, \ldots, \psi^{k_n}\tilde{\alpha}_n \rangle^X_{g_0,n,\beta_0} = \int_{[\overline{\mathcal{M}}_{g_0,n}(X,\beta_0)]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} ev_i^* \alpha_i, \tag{4.25}$$

where $[\overline{\mathcal{M}}_{g_0,n}(X,\beta_0)]^{vir}$ and the integration should be understood equivariantly. Notice

$$\begin{aligned} \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} deg(\tilde{\alpha}_i) &= \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} deg(\alpha_i) \\ &= dim[\overline{\mathcal{M}}_{g_0,n}(\mathbb{P}(V),\beta_0)]^{vir} \\ &= dim[\overline{\mathcal{M}}_{g_0,n}(X,\beta_0)]^{vir} - (1 - g_0 + (\beta_0, \mathcal{O}(1))) \\ &< dim[\overline{\mathcal{M}}_{g_0,n}(X,\beta_0)]^{vir}. \end{aligned} \tag{4.26}$$

Here we used the assumption that $(\beta_0, \mathcal{O}(1)) > g_0 - 1$. Integrating a lower degree equivariant cohomology class on a higher degree equivariant homology class results in 0, since there is no negative degree element in $H^*_{\mathbb{C}^*}(\{pt\})$. □

Recall that if $\sum_{i=1}^{n} k_i + \sum_{i=1}^{n} deg(\alpha_i) = dim[\overline{\mathcal{M}}_{g_0,n}(\mathbb{P}(V),\beta_0)]^{vir}$, by Lemma 1.2.2, we have

$$\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n \rangle^{X,tw,\mathcal{O}(1)}_{g,n,\beta} = \frac{\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n \rangle^X_{g,n,\beta}}{(-\lambda)^r}. \tag{4.27}$$

The $(-\lambda)$ on the denominator is due to the induced action of $\mathbb{C}^*$ on $\mathcal{O}(-1)$ that has weight -1 fiberwise. Therefore, under this degree condition of insertions, the leading term of the right-hand side of (4.23) is in fact an untwisted invariant.

Back to the goal of this section. Note that in the last line of (4.23), we sum over Γ with $\Gamma \neq \Gamma_0$. $E(\Gamma)$ has to be nonempty. Notice that the edge component must have nontrivial numerical class. As a result, for any $v \in V(\Gamma)$, we have $\beta_v < \beta_0$. Since $g_\Gamma = g_0$, we have $g_v \leq g_0$ for all $v \in V(\Gamma)$. Combining these observations together with Lemma 4.3.4, our induction can be achieved by applying (4.23) under the induction hypothesis and the degree condition of insertions in Lemma 4.3.4. In other words, (4.23) expresses $\langle \psi^{k_1}\alpha_i, \ldots, \psi^{k_n}\alpha_n \rangle^{\mathbb{P}(V)}_{g_0,n,\beta_0}$ in terms of Gromov–Witten invariants of the form $\langle \ldots \rangle^{\mathbb{P}(V)}_{g,n',\beta}$ with $\beta < \beta_0$ and $g \leq g_0$, plus the twisted invariants of S.

Example 4.3.5. If S is a point, it provides a way to compute $g = 0$ Gromov–Witten invariants of P^n from the ones of a point. In this case, any curve class is a fiber class in the sense of section 4.3.1. Note that our localization still works for fiber integrals when $g = 0$, but it fails when $g \geq 1$.

Let us apply our localization to compute $\langle\psi^{2n-1}\rangle^{\mathbb{P}^n}_{0,1,1}$. If we cheat by using the hypergeometric J-function, we know the answer is

$$\langle\psi^{2n-1}\rangle^{\mathbb{P}^n}_{0,1,1} = (-1)^n \binom{2n}{n}. \tag{4.28}$$

Now we apply master space technique and consider $\langle\psi^{2n-1}\rangle^{\mathbb{P}^{n+1}}_{0,1,1}$, which is apparently 0 due to virtual dimension. Let $\mathbb{C}^*$ act on $\mathbb{P}^{n+1}$ by sending $[x_0 : \ldots : x_{n+1}]$ to $[\lambda x_0 : \ldots : \lambda x_n : x_{n+1}]$. We have

$$\begin{aligned} 0 &= \langle\psi^{2n-1}\rangle^{\mathbb{P}^{n+1}}_{0,1,1} \\ &= \langle\psi^{2n-1}\rangle^{\mathbb{P}^n}_{0,1,1}\lambda^{-2} + \int_{\mathbb{P}^n} \frac{(\lambda - H)^{2n-1}(\lambda - H)}{\lambda^{n+1}(H-\lambda)} + \int_{\mathbb{P}^n} \frac{(-\lambda + H)^{2n-1}(-\lambda + H)}{\lambda^{n+1}(H-\lambda)} \\ &= \langle\psi^{2n-1}\rangle^{\mathbb{P}^n}_{0,1,1}\lambda^{-2} - 2(-1)^n\binom{2n-1}{n}\lambda^{-2}. \end{aligned} \tag{4.29}$$

Notice $2\binom{2n-1}{n} = \binom{2n}{n}$. We are done.

4.4 Proof of Theorem 4.1.1

Before the proof, let us state a few lemmas.

Lemma 4.4.1. *Let S be a smooth projective variety. Let V_1, V_2 be vector bundles over S such that $c(V_1) = c(V_2)$. Let $\mathbb{C}^*$ act on V_1, V_2 by scaling. We have the equality of twisted invariants*

$$\langle\psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n\rangle^{S,tw,V_1}_{g,n,\beta} = \langle\psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n\rangle^{S,tw,V_2}_{g,n,\beta} \in \mathbb{C}[\lambda, \lambda^{-1}]. \tag{4.30}$$

It has a similar appearance as the Theorem 4.1.1. But it is all about invariants on S twisted by two different vector bundles.

Proof. Since the $\mathbb{C}^*$ acts on V_i by scaling, we have

$$e_{\mathbb{C}^*}((V_i)_{g,n,\beta}) = \lambda^r + c_1((V_i)_{g,n,\beta})\lambda^{r-1} + \cdots + c_r((V_i)_{g,n,\beta}). \tag{4.31}$$

It suffices to prove that $c((V_i)_{g,n,\beta})$ depends only on the total Chern class $c(V_i)$ for $i = 1, 2$. But this can be seen using the Grothendieck Riemann–Roch formula

$$ch((V_i)_{g,n,\beta}) = (ft_{n+1})_* \left(ch(ev^*_{n+1}V_i) \cdot Td^\vee(\Omega_{ft_{n+1}})\right) \tag{4.32}$$

One can get more precise formulas by following the analysis in [8, Appendix 1] or [11]. Since $c(V_1) = c(V_2)$, we have $ch(V_1) = ch(V_2)$. And we readily have $ch(ev^*_{n+1}V_1) =$

$ch(ev_{n+1}^* V_2)$ by functoriality. The Todd class is determined by the moduli stacks $\overline{\mathcal{M}}_{0,n}(X,\beta)$ and its universal family, which is independent of V_1 and V_2. $\square$

Lemma 4.4.2. *Let S be a smooth projective variety. Let V be a vector bundle with $\mathbb{C}^*$ acting by scaling on fibers. Then*

$$\langle \psi^{k_1}\alpha_1, \dots, \psi^{k_n}\alpha_n \rangle_{g,n,\beta}^{S,tw,V}$$

can be determined by the Chern classes $c(V)$ and untwisted invariants of the form

$$\langle \psi^{k'_1}\alpha'_1, \dots, \psi^{k'_{n'}}\alpha'_{n'} \rangle_{g',n',\beta'}^{S},$$

where $g' \leq g$, $\beta' \leq \beta$.

Again it is a direct application of Grothendieck Riemann–Roch formula and we omit the details.

Remark 4.4.3. The actual determination of twisted invariants by untwisted invariants is very complicated. Besides direct Grothendieck Riemann–Roch calculation, the Quantum Riemann–Roch theorem in [8] may provide nice expressions under the formalism of quantized quadratic Hamiltonians. But in this paper, the precise algorithm is not needed.

The proof of 4.1.1 proceeds by applying Theorem 4.3.2 to both $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$. But there is still one subtlety. Now that the corresponding twisted invariants on S are identified, all the ingredients for the two cases in Theorem 4.3.2 now agree except the Mori cones $\mathrm{NE}(\mathbb{P}(V_1))$ and $\mathrm{NE}(\mathbb{P}(V_2))$. In general, the two Mori cones can be different. However, the condition in Lemma 4.3.4 is enough for (4.23) to provide relations between invariants on the projective bundle and twisted invariants. Lemma 4.3.4 requires a weaker condition on the curve class β_0 than $\pi_*\beta_0 \neq 0$ and β_0 being effective in $\mathbb{P}(V)$.

Fix an $i \in \{1,2\}$. Recall we identify $N_1(\mathbb{P}(V_i))$ with $N_1(\mathbb{P}(V_i \oplus \mathcal{O}))$ via push-forward under inclusion and we have $\mathrm{NE}(\mathbb{P}(V_i)) \subset \mathrm{NE}(\mathbb{P}(V_i \oplus \mathcal{O}))$ since push-forward preserves effectiveness. Also recall that by Assumption 1 in section 4.1.1, $\mathcal{O}_{\mathbb{P}(V_i)}(1)$ is ample on $\mathbb{P}(V_i)$. We prove a few lemmas under this assumption.

Lemma 4.4.4. *$\mathcal{O}_{\mathbb{P}(V_i \oplus \mathcal{O})}(1)$ is nef on $\mathbb{P}(V_i \oplus \mathcal{O})$.*

Proof. $\mathbb{P}(V_i)$ can be realized as the zero locus of a section s on the line bundle $\mathcal{O}_{\mathbb{P}(V_i \oplus \mathcal{O})}(1)$. Let C be an effective curve on $\mathbb{P}(V_i \oplus \mathcal{O})$. If $C \subset \mathbb{P}(V_i)$, $\mathcal{O}_{\mathbb{P}(V_i \oplus \mathcal{O})}(1)$ restricts to an ample line bundle on C by assumption. If otherwise, s does not vanish on the whole C. Therefore, $\mathcal{O}_{\mathbb{P}(V_i\oplus\mathcal{O})}(1)$ restricts to a line bundle of nonnegative degree as well. $\square$

Recall $V_i \subset \mathbb{P}(V_i \oplus \mathcal{O})$. Let $\iota_i : S \to \mathbb{P}(V_i \oplus \mathcal{O})$ be the inclusion given by the zero section of V_i, and $pr_i : \mathbb{P}(V_i \oplus \mathcal{O}) \to S$ the projection. Let $f_i \in \mathrm{NE}(\mathbb{P}(V_i \oplus \mathcal{O}))$ be the class of degree 1 curve on the fiber.

Lemma 4.4.5. *Any extremal curve class $\beta \in \mathrm{NE}(\mathbb{P}(V_i \oplus \mathcal{O}))$ with $(pr_i)_*\beta \neq 0$ must have $\beta = (\iota_i)_*(pr_i)_*\beta$.*

Proof. First of all, $\beta - (\iota_i)_*(pr_i)_*\beta = kf$ for some integer k. If $k > 0$, $\beta = kf + (\iota_i)_*(pr_i)_*\beta$ contradicts with β being extremal. On the other hand, notice $((\iota_i)_*(pr_i)_*\beta, \mathcal{O}_{\mathbb{P}(V_i\oplus\mathcal{O})}(1)) = 0$. We have

$$\begin{aligned} k =&(kf, \mathcal{O}_{\mathbb{P}(V_i\oplus\mathcal{O})}(1)) \\ =&(\beta - (\iota_i)_*(pr_i)_*\beta, \mathcal{O}_{\mathbb{P}(V_i\oplus\mathcal{O})}(1)) = (\beta, \mathcal{O}_{\mathbb{P}(V_i\oplus\mathcal{O})}(1)) \geq 0 \end{aligned} \tag{4.33}$$

since $\mathcal{O}_{\mathbb{P}(V_i\oplus\mathcal{O})}(1)$ is nef. The only case possible is $k = 0$. $\square$

Now $i \in \{1, 2\}$ is no longer fixed, and we are going to compare the $i = 1, 2$ cases. Recall $c(V_1) = c(V_2)$. We have the natural identification $\mathfrak{F}_{V_1\oplus\mathcal{O},V_2\oplus\mathcal{O}} : N_1(\mathbb{P}(V_1 \oplus \mathcal{O})) \cong N_1(\mathbb{P}(V_2 \oplus \mathcal{O}))$ from section 4.1.

Lemma 4.4.6. $\mathfrak{F}_{V_1\oplus\mathcal{O},V_2\oplus\mathcal{O}}(\mathrm{NE}(\mathbb{P}(V_1 \oplus \mathcal{O}))) = \mathrm{NE}(\mathbb{P}(V_2 \oplus \mathcal{O}))$.

Proof. Certainly $\mathfrak{F}_{V_1\oplus\mathcal{O},V_2\oplus\mathcal{O}}(f_1) = f_2$. One can also check that given a $\overline{\beta} \in \mathrm{NE}(S)$,

$$\mathfrak{F}_{V_1\oplus\mathcal{O},V_2\oplus\mathcal{O}}((\iota_1)_*\overline{\beta}) = (\iota_2)_*\overline{\beta}. \tag{4.34}$$

As a result, extremal rays of corresponding Mori cones are identified under $\mathfrak{F}_{V_1\oplus\mathcal{O},V_2\oplus\mathcal{O}}$. Thus, the Mori cones are identified as well. $\square$

Under the isomorphism $\mathfrak{F}_{V_1\oplus\mathcal{O},V_2\oplus\mathcal{O}}$, we won't distinguish $N_1(\mathbb{P}(V_i \oplus \mathcal{O}))$ and $\mathrm{NE}(\mathbb{P}(V_i \oplus \mathcal{O}))$ for different i. We are ready to apply the computation from section 4.3 to $V = V_1$ and $V = V_2$ together and compare the invariants. Before running the induction in section 4.3, the L_g are chosen so that $(\beta, \mathcal{O}_{\mathbb{P}(V_i)}(1) + L_g) > \max\{g - 1, 0\}$ for both $i = 1, 2$. Fix $g_0 \in \mathbb{N}$ and $\beta_0 \in \mathrm{NE}(\mathbb{P}(V_1)) \bigcup \mathrm{NE}(\mathbb{P}(V_2))$. Since Lemma 4.3.4 works under this condition, we can apply (4.23) to both $V = V_1$ and $V = V_2$ cases. We further put an induction hypothesis that invariants $\langle \cdots \rangle^{\mathbb{P}(V_i)}_{g,n',\beta}$ with $g \le g_0$ and $\beta < \beta_0$ are identified as in Theorem 4.1.1 for $i = 1, 2$. Now the graph sums in (4.23) starting with $\sum_{\Gamma \neq \Gamma_0} \frac{1}{Aut(\Gamma)} \ldots$ are the same for $V = V_1$ and $V = V_2$ by induction and Lemma 4.4.1. Thus Theorem 4.1.1 is proven by induction.

4.5 Application to the blow-ups

Theorem 4.1.1 can be applied to blow-ups at smooth centers to imply the following

Theorem 4.5.1. *Let $Z \subset Y$ be inclusions of smooth projective varieties and $N_{Z/Y}$ be the normal bundle of Z. Let $\tilde{Y}$ be the blow-up of Y at Z and E be the exceptional divisor. The absolute Gromov–Witten invariants of $\tilde{Y}$ can be determined by the absolute Gromov–Witten invariants of Y and Z, plus the following topological data:*

1. *The cohomology rings $H^*(Y)$, $H^*(Z)$ and their pull-back map under inclusion.*

2. *The Chern classes $c_i(N_{Z/Y}) \in H^*(Y)$.*

In fact [16, Theorem 5.15] plus degeneration formula already implies a similar determination of Gromov–Witten invariants that requires some additional information. To be precise, [16, Theorem 5.15] and degeneration formula implies

Theorem 4.5.2. *Under the same set-up of Theorem 4.5.1. The absolute Gromov–Witten invariants of $\tilde{Y}$ can be determined by the absolute Gromov–Witten invariants of Y and Z, plus the following data:*

1. *The cohomology rings $H^*(Y)$, $H^*(Z)$ and their pull-back map under inclusion.*

2. *The Chern classes $c_i(N_{Z/Y}) \in H^*(Y)$.*

3. *The absolute Gromov–Witten invariants of $\mathbb{P}(N_{Z/Y} \oplus \mathcal{O})$*

Apparently, our Theorem 4.1.1 adds to this known result by saying the invariants of $\mathbb{P}(N_{Z/Y} \oplus \mathcal{O})$ can already be determined by the invariants of Z and Chern classes $c_i(N_{Z/Y})$, and thus the requirement (c) in Theorem 4.5.2 is redundant. The rest of the section is a brief explanation that [16, Theorem 5.15] + degeneration formula implies Theorem 4.5.2.

First of all, by [27], (3) is enough to determine all relative invariants of the pair $(\mathbb{P}(N_{Z/Y} \oplus \mathcal{O}), \mathbb{P}(N_{Z/Y}))$. Let us recall the main theorem in [16]. In view of [16, Definition 5.6], its content can be rephrased as the following.

Theorem 4.5.3. *Absolute Gromov–Witten invariants of Y and (1), (2), (3) in Theorem 4.5.2 are enough to determine relative Gromov–Witten invariants of the following form (relative invariants of standard weighted relative graphs in [16]):*

$$\langle p^*\sigma_1, \ldots, p^*\sigma_n | \mu \rangle_{g,\beta}^{\tilde{Y},E}$$

where $p : \tilde{Y} \to Y$ is the contraction.

Although insertions are pull-backs from Y, these relative invariants are enough to determine the absolute Gromov–Witten invariants of $\tilde{Y}$. Applying deformation to the normal cone, we form $\mathfrak{X} = Bl_{E\times\{0\}}\tilde{Y} \times \mathbb{A}^1$. The fiber at 0 is a union of Y and $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O})$ glued along E. Let

$$\iota_1 : \tilde{Y} \to \mathfrak{X}, \quad \iota_2 : \mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \to \mathfrak{X}$$

be the embeddings of corresponding irreducible components to the fiber at 0. An absolute invariant of $\tilde{Y}$ can thus be computed by relative invariants of the pairs $(\tilde{Y}, E)$ and $(\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}), E)$.

During this process, one has different choices for the insertions of the relative invariants. Given $\bar{\alpha} \in H^*(\tilde{Y})$ an insertion of the absolute invariant of $\tilde{Y}$, to apply degeneration formula, one needs to find a lifting $\alpha \in H^*(\mathfrak{X})$ whose restriction on a general fiber is $\bar{\alpha}$. The corresponding insertions for relative invariants of either $(\tilde{Y}, E)$ or $(\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}), E)$ are $\iota_1^*\alpha$ or $\iota_2^*\alpha$, respectively. One can make an obvious choice by finding the α such that $\iota_1^*\alpha = \bar{\alpha}$ and $\iota_2^*\alpha = \pi^*(\bar{\alpha}|_E)$ where $\pi : \mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \to E$. However, some flexibility exists. One can add to this α a cohomology class whose Poincaré dual has support in E. Thus we have

Lemma 4.5.4. *Given an $\bar{\alpha} \in H^*(\tilde{Y})$, there exists a lifting $\alpha' \in H^*(\mathfrak{X})$ such that*

$$\begin{aligned} \iota_1^*\alpha &= \bar{\alpha} + (\iota_E)_!\sigma, \\ \iota_2^*\alpha &= \pi^*(\bar{\alpha}|_E) - h\pi^*\sigma \end{aligned} \tag{4.35}$$

where $\iota_E : E \to \tilde{Y}$ is the inclusion, $\sigma \in H^*(E)$ and $h = c_1(\mathcal{O}_{\mathbb{P}_E(\mathcal{O}(-1)\oplus\mathcal{O})}(1))$.

The goal of this section is done if we can prove there exists a σ such that $\bar{\alpha} + (\iota_E)_!\sigma$ is the pull-back of a certain class along p. But notice that $\bar{\alpha} - p^*p_!\bar{\alpha} = (\iota_E)_!\sigma'$ for some $\sigma' \in H^*(E)$. Choosing $\sigma = \sigma'$ is enough. Therefore, Theorem 4.5.2 is proven.

REFERENCES

[1] M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology, 23 (1984), pp. 1–28.

[2] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math., 128 (1997), pp. 45–88.

[3] K. Behrend and M. Y., *Stacks of stable maps and gromov–witten invariants*, Duke Math. J., 85 (1996), pp. 1–60.

[4] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math., 98 (1973).

[5] J. Brown, *Gromov-Witten Invariants of Toric Fibrations*, (2009). `arXiv:0901.1290` version.

[6] I. Ciocan-Fontanine and B. Kim, *Wall-crossing in genus zero quasimap theory and mirror maps*, Algebraic Geom., 1 (2014), pp. 400–448.

[7] T. Coates, A. Corti, H. Iritani, and H.-H. Tseng, *Computing Genus-Zero Twisted Gromov-Witten Invariants*, Duke Math. J., 147 (2009), pp. 377–438.

[8] T. Coates and A. Givental, *Quantum Riemann–Roch, Lefschetz and Serre*, Ann. of Math., 165 (2007), pp. 15–53.

[9] T. Coates, A. Givental, and H.-H. Tseng, *Virasoro constraints for toric bundles*, (2015). `arXiv:1508.06282`.

[10] D. A. Cox, J. B. Little, and H. Schenck, *Toric Varieties*, (2011).

[11] C. Faber and R. Pandharipande, *Hodge integrals and gromov-witten theory*, Invent. Math., 139 (2000), pp. 173–199.

[12] W. Fulton, *Introduction to Toric Varieties*, (1993).

[13] ——, *Intersection Theory*, (1998).

[14] A. Givental, *Symplectic geometry of frobenius structures*, (2003). `arXiv:math/0305409`.

[15] T. Graber and R. Pandharipande, *Localization of virtual classes*, (1997). `arXiv:math/9708001`.

[16] J. Hu, T. Li, and Y. Ruan, *Birational cobordism invariance of uniruled symplectic manifolds*, Invent. Math., 172 (2008), pp. 231–275.

[17] Y. Jiang, *The orbifold cohomology ring of simplicial toric stack bundles*, Illinois J. Math., 52 (2008), pp. 493–514.

[18] Y.-P. Lee, H.-W. Lin, F. Qu, and C.-L. Wang, *Invariance of quantum rings under ordinary flops: III*, (2014). arXiv:1401.7097.

[19] Y.-P. Lee, H.-W. Lin, and C.-L. Wang, *Invariance of quantum rings under ordinary flops: I*, (2011). arXiv:1109.5540v3.

[20] ——, *Invariance of quantum rings under ordinary flops: II*, (2013). arXiv:math/1311.5725.

[21] ——, *Birational Transformation and degeneration in Gromov–Witten theory, reported by the first author in the AMS summer institute*, (2015).

[22] ——, *Quantum cohomology under birational maps and transitions*, (2016). submitted to Proceedings of String-Math 2015 conference.

[23] Y.-P. Lee and R. Pandharipande, *A Reconstruction Theorem in Quantum Cohomology and Quantum K-Theory*, Amer. J. Math., 126 (2004), pp. 1367–1379.

[24] ——, *Frobenius manifolds, Gromov–Witten theory, and Virasoro constraints*, (2004). unfinished book available at https://www.math.utah.edu/ yplee/research/ .

[25] J. Li and G. Tian, *Virtual moduli cycles and gromov–witten invariants of algebraic varieties*, J. Amer. Math. Soc., 11 (1998), pp. 119–174.

[26] C.-C. M. Liu, *Localization in Gromov–Witten Theory and Orbifold Gromov-Witten Theory*, Handbook of moduli, II (2013).

[27] D. Maulik and R. Pandharipande, *A topological view of Gromov–Witten theory*, Topology, 45 (2006), pp. 887–918.

[28] A. Mustata and A. Mustata, *Gromov–Witten invariants for varieties with C* action*, (2015). arXiv:1505.01471.

[29] S. Payne, *Moduli of toric vector bundles*, Compos. Math., 144 (2008), pp. 1199–1213.

[30] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ., 14 (1974).